Tutorial 2

(1)

$$\frac{\dot{a}^2}{a^2} = H_0^2 \left[\Omega_\Lambda + \Omega_{nr}\left(\frac{a_0}{a}\right)^3\right]$$

$$y = \frac{a}{a_0}, \quad \& \quad x = tH_0$$

$$\Rightarrow \frac{1}{y^2}\left(\frac{dy}{dx}\right)^2 = \left[\Omega_\Lambda + \frac{\Omega_{nr}}{y^3}\right]$$

$$\frac{dy}{y} \cdot \frac{1}{\left[\Omega_\Lambda + \frac{\Omega_{nr}}{y^3}\right]^{1/2}} = dx$$

$$\frac{y^{1/2}\,dy}{\Omega_{nr}^{1/2}\left[1 + \frac{\Omega_\Lambda}{\Omega_{nr}} y^3\right]^{1/2}} = dx$$

Substitute :

$$\sinh^2\frac{\eta}{2} = \frac{\Omega_\Lambda}{\Omega_{nr}} y^3$$

$$\Rightarrow \cancel{\sinh\frac{\eta}{2}} \cosh\frac{\eta}{2}\, d\eta = \frac{3\Omega_\Lambda}{\Omega_{nr}} y^2\, dy$$

$$= 3\cancel{\left(\frac{\Omega_\Lambda}{\Omega_{nr}}\right)^{1/2} y^{3/2}} \cdot \left(\frac{\Omega_\Lambda}{\Omega_{nr}}\right)^{1/2} y^{1/2}\, dy$$

$$\Rightarrow \quad y^{1/2}\, dy = \frac{1}{3}\left(\frac{\Omega_{m}}{\Omega_\Lambda}\right)^{1/2} \cosh\frac{\eta}{2}\, d\eta$$

$$\Rightarrow \quad dx = \frac{1}{3}\,\frac{1}{\Omega_\Lambda^{1/2}}\cdot d\eta$$

$$\Rightarrow \quad tH_0 = \frac{1}{3}\,\frac{1}{\Omega_\Lambda^{1/2}}\cdot \eta \quad + \text{Const.}$$

$$\Rightarrow \quad \frac{\eta}{2} = \frac{3\,\Omega_\Lambda^{1/2}\, tH_0}{2} + \text{Const.}$$

$$\Rightarrow \quad \sinh\left(\frac{\eta}{2}\right) = \left(\frac{\Omega_\Lambda}{\Omega_{m}}\right)^{1/2}\left(\frac{a}{a_0}\right)^{3/2} = \sinh\left[\frac{3\,\Omega_\Lambda^{1/2}\, tH_0}{2} + \text{Const}\right]$$

$a = 0 \;@\; t = 0$

$\Rightarrow \text{Const} = 0$

$$\Rightarrow \quad \boxed{a = a_0 \left(\frac{\Omega_{m}}{\Omega_\Lambda}\right)^{1/3} \sinh^{2/3}\left[\frac{3\,\Omega_\Lambda^{1/2}\, tH_0}{2}\right]}$$

(2) $$\lim_{t\to 0} \frac{a}{a_0} = \left(\frac{\Omega_{m,0}}{\Omega_\Lambda}\right)^{1/3} \left[\sinh\left(\frac{3\,\Omega_\Lambda^{1/2}}{2} t H_0\right)\right]^{2/3}$$

$$= \left(\frac{\Omega_{m,0}}{\Omega_\Lambda}\right)^{1/3} \cdot \left[\frac{3\,\Omega_\Lambda^{1/2}}{2} t H_0\right]^{2/3}$$

as $\sinh\theta \to \theta$ at small θ.

$$\Rightarrow \lim_{t\to 0} \frac{a}{a_0} = \Omega_{m,0}^{1/3}\left(\frac{3H_0 t}{2}\right)^{2/3} \propto t^{2/3}$$

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(3) $$\lim_{t\to\infty} \frac{a}{a_0} = \left(\frac{\Omega_{m,0}}{\Omega_\Lambda}\right)^{1/3} \left[\sinh\left(\frac{3\,\Omega_\Lambda^{1/2}}{2} t H_0\right)\right]^{2/3}$$

$$\simeq \left(\frac{\Omega_{m,0}}{\Omega_\Lambda}\right)^{1/3} \left[\frac{1}{2} \exp\left(\frac{3\,\Omega_\Lambda^{1/2}}{2} t H_0\right)\right]^{2/3}$$

$$= \left(\frac{\Omega_{m,0}}{4\,\Omega_\Lambda}\right) \exp\left(\Omega_\Lambda^{1/2} t H_0\right)$$

(4) $$\frac{8\pi G}{3}\rho_\Lambda = H_0^2 \Omega_\Lambda$$

$$\frac{8\pi G}{3}\rho_{nr} = H_0^2 \Omega_{nr} (1+z)^3$$

$$\Rightarrow \frac{\rho_{nr}}{\rho_\Lambda} = \frac{\Omega_{nr}}{\Omega_\Lambda}(1+z)^3 = 1$$ at equality.

$$\Rightarrow z_{eq} = \left(\frac{\Omega_\Lambda}{\Omega_{nr}}\right)^{1/3} - 1 = \left(\frac{0.7}{0.3}\right)^{1/3} - 1$$

$$= 0.33$$

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(5) $$\frac{\ddot{a}}{a} = H_0^2 \left[-\frac{1}{2}\Omega_{nr}(1+z)^3 + \Omega_\Lambda \right]$$

This is negative at large z. It crosses zero at

$$-\frac{1}{2}\Omega_{nr}(1+z)^3 + \Omega_\Lambda = 0$$

$$\Rightarrow (1+z)^3 = \frac{2\Omega_\Lambda}{\Omega_{nr}}$$

$$\Rightarrow z = \left(\frac{2\Omega_\Lambda}{\Omega_{nr}}\right)^{1/3} - 1 = \left(\frac{1.4}{3}\right)^{1/3} - 1$$

$$= 0.67$$

$\Rightarrow$ expansion begins to accelerate at $z = 0.67$ in this model.