# Tutorial 1

(1)

$$\phi = \frac{1}{2} v_c^2 \ln\left[R_c^2 + R^2\right]$$

$$\rho = \frac{1}{4\pi G} \nabla^2 \phi = \frac{1}{4\pi G} \frac{1}{R^2} \frac{\partial}{\partial R}\left(R^2 \frac{\partial \phi}{\partial R}\right)$$

$$\frac{\partial \phi}{\partial R} = \frac{1}{2} v_c^2 \frac{\partial}{\partial R} \ln\left(R_c^2 + R^2\right)$$

$$= \frac{1}{2} v_c^2 \frac{2R}{R_c^2 + R^2}$$

$$\Rightarrow \rho = \frac{v_c^2}{4\pi G} \frac{1}{R^2} \cdot \frac{\partial}{\partial R}\left[\frac{R^3}{R_c^2 + R^2}\right]$$

$$= \frac{v_c^2}{4\pi G} \cdot \frac{1}{R^2}\left[\frac{3R^2}{R_c^2 + R^2} - \frac{2R^4}{(R_c^2 + R^2)^2}\right]$$

$$= \frac{v_c^2}{4\pi G}\left[\frac{3(R_c^2 + R^2) - 2R^2}{(R_c^2 + R^2)^2}\right]$$

$$= \frac{v_c^2}{4\pi G}\left[\frac{3R_c^2 + R^2}{(R_c^2 + R^2)^2}\right]$$

$$\lim_{R \to 0} \rho = \frac{v_c^2}{4\pi G}\left[\frac{3R_c^2 + R^2}{R_c^4\left(1 + \frac{R^2}{R_c^2}\right)^2}\right]$$

$$= \frac{v_c^2}{4\pi G R_c^4}\left[(3R_c^2 + R^2)\left(1 - \frac{2R^2}{R_c^2}\right)\right]$$

$$\Rightarrow \lim_{R\to 0} \rho = \frac{v_c^2}{4\pi G R_c^4}\left[3R_c^2 + R^2 - 6R^2 + O(R^4)\right]$$

$$= \frac{3v_c^2}{4\pi G R_c^2}\left[1 - \frac{5}{3}\frac{R^2}{R_c^2}\right]$$

$$\lim_{R>>R_c} \rho = \frac{v_c^2}{4\pi G R^2}\left[\left(1 + 3R_c^2/R^2\right)\left(1 + \frac{R_c^2}{R^2}\right)^{-2}\right]$$

$$\simeq \frac{v_c^2}{4\pi G R^2}\left[\left(1 + \frac{3R_c^2}{R^2}\right)\left(1 - \frac{2R_c^2}{R^2}\right)\right]$$

$$= \frac{v_c^2}{4\pi G R^2}\left[1 + \frac{R_c^2}{R^2} + O\left(\frac{R_c^4}{R^4}\right)\right]$$

---

(2) $$\frac{v_{circ}^2}{R} = \frac{GM(R)}{R^2} = \frac{\partial \Phi}{\partial R}$$

$$= v_c^2 \frac{R}{R^2 + R_c^2}$$

$$\Rightarrow v_{circ}^2 = \frac{v_c^2}{1 + R_c^2/R^2} = \frac{v_c^2 R^2}{R^2 + R_c^2}$$

$$\Rightarrow v_{circ} = \frac{v_c R}{R + R_c}$$

$$\lim_{R\to 0} v_{circ} = 0$$

$$\lim_{R>>R_c} v_{circ} = v_c$$