Solutions to Assignment 2

1. A dice is rolled three times. What is the probability that each time the number obtained is larger than the previous number obtained? Is this higher the probability that the number obtained each time is smaller than the previous one?

Solution: The sequence (a, b, c) of faces that appear must be in increasing order. Let us denote

$$A = \{(a, b, c) | a < b < c \text{ and } 1 \leq a, b, c \leq 6\}$$

The probability of each sequence is $(1/6)^3$ (since each face is equally likely). Thus the probability is $|A|/6$.

This leaves the problem of counting the number $|A|$ of elements in A. This can be done in many ways including "by hand"! One way is to note that if we choose a subset of 3 elements of $[1, 6]$, then it can be *ordered* in a unique way. So the number of elements in A is the *same* as the number of subsets of $[1, 6]$ with 3 elements. The latter number is $\binom{6}{3} = 20$.

So the probability is $20/216 = 5/54$. Note that the same sequence is decreasing in reverse time and since the probability does not depend on the order of events (since they are independent!) so the decreasing sequence has the same probability!

2. Player A rolls two dice with the hope of getting at least one six. Player B rolls four dice with the hope of getting at least two sixes. Which player has a better probability of success?

Solution: Since we are only look at getting a 6 or not a six, we can think of the problem using the biased coin(!) with probability of $p = 1/6$ of getting a 6 and $1 - p = 5/6$ of *not* getting 6. The probability of getting at least one 6 with two dice is

$$\binom{2}{1} p^1 (1 - p)^1 + \binom{2}{2} p^2 = 2 \cdot (1/6) \cdot (5/6) + (1/6)^2 = 11/36$$

The probability of getting at least two 6's with four dice is

$$\binom{4}{2} p^2 (1-p)^2 + \binom{4}{3} p^3 (1-p) + \binom{4}{4} p^4 = 6 \cdot (1/6)^2 \cdot (5/6)^2 + 4 \cdot (1/6)^3 \cdot (5/6) \cdot (1/6)^4 = 171/1296$$

The second probability is much smaller than the first!

3. Two players flip 9 coins. Player A wins if the number of Heads is Even, Player B wins if the number of Heads is Odd. Which one has a better chance? Does this change if there are 10 coins?

Solution: We apply the binomial distribution to get the probability of an even number of Heads

$$\frac{\sum_{r=0}^{4}\binom{9}{2r}}{2^9}$$

The probability of an odd number of Heads is

$$\frac{\sum_{r=0}^{4}\binom{9}{2r+1}}{2^9}$$

Now $\binom{9}{2r+1} = \binom{9}{9-(2r+1)} = \binom{9}{2(4-r)}$ so we see that both the numbers are the same! We could also have argued "by symmetry" that Heads and Tails are equally likely. An even number of Heads means an odd number of Tails (since 9 is odd!) and so both probabilities are equal.

The above argument does not work for 10 coins. We need to compare

$$\sum_{r=0}^{5}\binom{10}{2r} \text{ and } \sum_{r=0}^{4}\binom{10}{2r+1}$$

The first one has more terms, so we may expect it to be larger. However, let us calculate using binomial theorem.

$$0 = (-1+1)^{10} = \sum_{r=0}^{5}\binom{10}{r}(-1)^r 1^{10-r} = \sum_{r=0}^{5}\binom{10}{2r} - \sum_{r=0}^{4}\binom{10}{2r+1}$$

So in fact these numbers are equal in this case as well! The probabilities are the same in both cases and so they are 1/2!

4. A bus starts with 12 passengers and makes 6 stops and there are no passengers left. What is the probability that (a) all passengers got off at the same stop (b) that the same number of passengers got off at each stop.

Solution: We make the assumption (based on lack of information!) that it is equally likely for a passenger to choose any particular stop out of the 6 stops. Thus, the probability of a passenger picking a particular stop is 1/6. Moreover, we also assume that the choices made by the passengers are independent of each other.

So this is like rolling 12 dice.

The probability of all dice showing the same number is $6 \cdot (1/6)^{12} = (1/6)^{11}$.

The probability that the same number of passengers got off each each stop means that 2 passengers got off at each stop. This is The same as the probability that the numbers on the dice appear in pairs. We can calculate using the multinomial distribution that the probability is

$$\binom{12}{2,2,2,2,2,2}(1/6)^12$$

5. Four digits (0 to 9) are chosen at random in order to make a 4-digit number. What is the probability that the number is divisible by 5? What is the probability that the number is divisible by 3?

Solution: The choice "at random" is taken to mean that each digit is equally likely and that the choices are independent of each other. Thus, the probability of choosing any particular digit is 1/10. So this is like rolling four 10-sided dice.

The number is divisible by 5 if the last digit is 0 or 5. Thus the probability does not depend on the previous digits and is just $1/10 + 1/10 = 1/5$.

To do the divisibility by 3 problem, we need to calculate the number of elements of the set A

$$A = \{(a, b, c, d)|a, b, c, d \in [0, 9] \text{ and } 3|(a + b + c + d)\}$$

This is *not* trivial but students can try it out!

This problem is proof that thinking "smartly" sometimes put you on the wrong track!

The better approach is to just note that each number between 0000 and 9999 has probability 1/10000. So we only need to *count* those which are divisible by 3. This is all multiples of numbers between 0000 and 3333; in other words 3334 of them. The answer is therefore 3334/10000.