Random Variables

In most applications of the Law of infinity that use events, we only need to use two fundamental rules:

- If $x < 1$ then $\inf x^n = 0$.
- If $x < 1$ then $\sum_n x^n = 1/(1 - x)$.

To make greater use of the infinite in computing probabilities, we need to make use of the notion of random variables as given below.

Random Variables

Another way of dealing with infinity is to use "variables". So far, we have described probability in terms of events. How are events demarcated? The answer is "by values of random variables."

In general, a random variable can take values anywhere, in integers, in real numbers, in a group and so on. The value of a random variable represents the result of an experiment. In most cases, in this course we will deal with real-valued random variables.

For example, we have a random variable X associated with a coin flip which takes two values 0 for Tail and 1 for Head; so the event of Head is the same as $X = 1$.

However, random variables are really most useful when the possible values are infinite. Consider the experiment where we repeatedly flip an unbiased coin until we get a Head. Let W be the random variable that counts the number of flips needed. The event $W = n$ is the same as $T_1 \cap \cdots \cap T_{n-1} H_n$ where T_i represents Tail on the i-flip and H_i represents Head on the i-flip. Thus, $P(W = n) = 1/2^n$. More generally, if we have a biased coin with probability p of Head and $1 - p$ for Tail, then $P(W = n) = p(1 - p)^{n-1}$.

When a random variable X takes values as real numbers, we can define events such as $X \leq t$. This gives us a function, $F_X(t) = P(X \leq t)$ which is usually called the (cumulative) distribution function of X. We note that for $t < s$ we have $(X \leq t) \subset (X \leq t)$ and thus $F_X(t) \leq F_X(s)$. In other words, F_X is a monotonically increasing (non-decreasing) function of t that limits to 0 as $t \to -\infty$ and limits to 1 as $t \to \infty$. One can also show that it is right continuous. Conversely, any such function can be seen to be the distribution function for a random variable.

For example, the distribution function for the random variable W considered above is given by $F_W(t) = 1 - (1 - p)^n$ where $n = \lfloor t \rfloor$ is the integer part of t when $n \geq 0$ and $F_W(t) = 0$ for $t < 1$. If we flip a coin each second, then this represents the probability that we will wait for at least t seconds before seeing a Head.

Whatever algebraic rules exist for combinations of the values of random variables also exist for combinations of the random variables themselves. Specifically, if X, Y are real-valued random variables, then we can form cX (for a real number c), $X + Y$, $X \cdot Y$, $\min X, Y$ and so on. If $P(Y = 0) = 0$, then we also have X/Y. There is also a "constant" random variable c which takes the value c with probability 1 and any other value with probability 0.

Another example, is that of the distribution function for the random variable S_k that counts the number of Heads in k flips. If X_i represents the random variable (with value 0 or 1) associated with the i-th flip, the we see that $S_k = X_1 + X_2 + \cdots + X_k$. This example can be used to understand that F_{X+Y} is *not* $F_X + F_Y$; in fact, the latter is *not* the distribution function of a random variable since it has the limit 2 as $t \to \infty$!

The following limiting construction of a random variable is of great importance in many applications. If Y is a random variable and X_n is a sequence of random variables so that $X_n \leq Y$ with probability 1, then $X = \sup X_n$ is also a random variable and $X \leq Y$. Once one can talk about sup we can also talk about limits in general under suitable assumptions of bounded-ness.

For example, consider X_n to be the random variable associated with the scaled random walk of n steps where the step size is halved every time. One can show that the limit of these random variables is U the random variable which is uniformly distributed in $[-1, 1]$. In other words,

$$F_U(t) = \begin{cases} 0 & \text{for } t < -1 \\ (t+1)/2 & \text{for } t \in [-1, 1] \\ 1 & \text{for } t > 1 \end{cases}$$

In general, the graph of the distribution function is shaped like the letter 'S'. There could be "jumps" like in a staircase and "continuous climbs" like the above case. In addition, one could have "fractal" phenomena like the "devil's staircase" constructed by Minkowski!

Our further study of probability will be based on random variables and the events defined by restrictions on the values of random variables.