Infinity

One of the qualifications of a mathematician is that she/he must master the concept of the infinite (or at least the incredibly large!). So far we have worked with the following laws of probability:

- The probability of the "empty" event is 0, i. e. $P(\emptyset) = 0$.
- If $A \subset B$ then $P(A) \leq P(B)$.
- The "excluded middle" law, $P(A) + P(A^c) = 1$.
- The addition law $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

We have use Ω to denote the "universe" of all possible outcomes of an experiment; it is also called the "sample space" and we have $\Omega = (\emptyset)^c$ and $P(\Omega) = 1$.

We have the notion of monotone (and increasing) sequence A_n of events $A_1 \subset A_2 \subset \ldots$. The "union" of these events is the event A defined by the condition that at least one of these events occurs. Note that this means that all "larger" events also occur by our interpretation of containment in the sense of events. By the laws of probability this gives $P(A_1) \leq P(A_2) \leq \ldots$, an increasing (non-decreasing) sequence of numbers all of which are ≤ 1. Thus, we have a least-upper-bound. One additional law of probability that seems entirely reasonable is:

- Given an increasing sequence $A_1 \subset A_2 \subset \ldots$ of events with union A, the least-upper-bound (supremum or "sup") of $P(A_n)$ is $P(A)$.

Variants and Justifications

Reversing this law by using complements and noting that the greatest-lower-bound (infimum or "inf") of $1 - P(A_n)$ is $1 - \sup_n P(A_n)$, we can also formulate this law as follows.

- Given a decreasing sequence $A_1 \supset A_2 \supset \ldots$ of events with intersection A, the greatest-upper-bound (infimum or "inf") of $P(A_n)$ is $P(A)$.

A very useful case of this is where $A_1 = B_1$, $A_2 = B_1 \cap B_2$ and so on $A_n = B_1 \cap \cdots \cap B_n$. In this case A is also the intersection of all the B_n's.

A particular case of this (which can also be used to give a justification for the above law!) is to consider the possibility of an "endless" collection of Heads for unbiased independent coin flips. Let H_n be the event of Head on the n flip and $A_n = H_1 \cap \cdots \cap H_n$. Then A_n is the event that all the flips numbered 1 to n result in Heads. The probabability $P(A_n)$ is clearly $1/2^n$. The event A is the event of *all* flips resulting in Heads. By the law given above $P(A) = \inf_n(1/2^n) = 0$.

As a paradoxical example, one can extend the above to see that any *given* sequence of Heads and Tails has probability 0! However, this does not actually contradict anything since no one can flip infnitely many coins—as my Physics teacher once said "There are no infinities in reality!"

Repeatable Experiments

This can be extended to a biased coin or more generally to any sequence of repeatable experiments. If an experimental result (for a single experiment) is an event B with probability p, we say that the experiment is repeatable if the probability of the "same" event in the n-th experiment has probability p as well. Let is denote the occurence of this event in the n-th experiment as B_n.

For the experiment to have a probabilistic result, we should assume that $p < 1$. So what is the probability that the result of the experiment every time is B? This is the probability that all the B_n occur. Now if $A_n = B_1 \cap \cdots \cap B_N$, then by the independence of the experiments we have $p(A_n) = p \cdot p \cdots p = p^n$. By the above low of probability, we see that $P(\cap_n B_n) = \inf P(A_n) = \inf p^n = 0$!

This is why, when a certain event is seen every time we conduct an experiment for a large number of times, we are convinced that this *always* happens. In other words, we are convinced that the probability is 1!

At the same time, we can also look at the case where $P(B_n) = p > 0$ is *very small*, and consider the possiblity that it *never* happens. This is the case where $A_n = B_1^c \cap B_2^c \cap \ldots B_n^c$. We again see that the probability that the event never happens is $P(\cap_n B_n^c) = \inf P(A_n) = \inf(1-p)^n = 0$. Thus, we can assert that even events with low probability, do eventually happen if the same experiment is (independently) repeated sufficiently often!