Solutions to Assignment 1

1. Which of the following are correct identities for subsets A, B, C of a fixed set S? Provide examples if they are not correct or prove using Boole's laws if they are correct.

$$\begin{aligned}(A\cup B)-C &= A\cup(B-C)\\ (A\cap B)\cap C &= (A\cap B)\cap(C\cup B)\\ A\cup B &= (A\setminus A\cap B)\cup B\\ (A\cap B)\cup(B\cap C)\cup(C\cap A) &\supset A\cap B\cap C\\ (A\cup B)^c\cap C &= (A^c\cap C)\cap(B^c\cap C)\\ A\cap B^c\cap C &\subset A\cup B\end{aligned}$$

Solution: One can use Venn diagrams or with "truth tables" (replacing $\cap$ with $\wedge$, $\cup$ with $\vee$, $A\subset B$ with $\neg A\vee B$. and $A-B$ with $A\wedge\neg B$.

1. From the following truth table

A	B	C	$A\cup B$	$(A\cup B)-C$	$B-C$	$A\cup(B-C)$	$(A\cup B)-C$ $=A\cup(B-C)$
1	1	1	1	0	0	1	0
1	1	0	1	1	1	1	1
1	0	1	1	0	0	1	0
1	0	0	1	1	0	1	1
0	1	1	1	0	0	0	1
0	1	0	1	1	1	1	1
0	0	1	0	0	0	0	1
0	0	0	0	0	0	0	1

we see that the two are different.

2. From the following truth table

A	B	C	$A\cap B$	$(A\cap B)\cap C$	$C\cup B$	$(A\cap B)\cap(C\cup B)$	$(A\cap B)\cap C$ $=(A\cap B)\cap(C\cup B)$
1	1	1	1	1	1	1	1
1	1	0	1	0	1	1	0
1	0	1	0	0	1	0	1
1	0	0	0	0	0	0	1
0	1	1	0	0	1	0	1
0	1	0	0	0	1	0	1
0	0	1	0	0	1	0	1
0	0	0	0	0	0	0	1

we see that the two are different.

3. From the following truth table

A	B	$A \cup B$	$(A \cap B)$	$A - (A \cap B)$	$(A - (A \cap B)) \cup B$	$(A \cup B) = (A - (A \cap B)) \cup B$
1	1	1	1	0	1	1
1	0	1	0	1	1	1
0	1	1	0	0	1	1
0	0	0	0	0	0	1

we see that the two are the same.

4. From the following truth table

A	B	C	$A \cap B$	$(B \cap C)$	$C \cap A$	$(A \cap B) \cup (B \cap C) \cup (C \cap A)$	$A \cap B \cap C$	$(A \cap B) \cup (B \cap C) \cup (C \cap A) \supset A \cap B \cap C$
1	1	1	1	1	1	1	1	1
1	1	0	1	0	0	1	0	1
1	0	1	0	0	1	1	0	1
1	0	0	0	0	0	0	0	1
0	1	1	0	1	0	1	0	1
0	1	0	0	0	0	0	0	1
0	0	1	0	0	0	0	0	1
0	0	0	0	0	0	0	0	1

we see that the relation holds.

Similar truth tables can be used for the remaining problems.

2. Assume that a class has students from each part of India (North/South/East/West), comprising of Boys and Girls, so that each of the letters of the English alphabet is the starting letter for exactly one student of each sex from each region. In other words, the information (Region, Sex, Starting Letter) uniquely specifies a student. Let A be the property that a student is from the North, B be the property that the student is a Boy, C be the property that the student's name starts the letter 'C' and D be the property that the student plays basketball for IISER Mohali. Assume that a student is chosen "at random" from the class. Explain the meaning of each event (set) in the list below:

$$D^c; A \cap B; A^c \cap B^c \cap D^c; D \setminus C$$

Solution:

1. The chosen student does not play basketball.
2. The chosen student is a Boy from the North.
3. The chosen student is not from the North, is a Girl and does not play Basketball.
4. The chosen student plays Basketball and the name does not start with 'C'.

3. Consider the set S of students in IISER Mohali. We pick a student at random. Someone asserts that the probability that the student is from Hostel 7 is 0.25, and that the probability that it is a Girl student is 0.5, and the probability that the student's name starts with 'Z' is 0.05. Finally, the person also says that the probability that is not a Girl student and not from Hostel 7 and that the name starts with A is 0.7. Is it possible that all four estimates of probability are correct? If not, why not?

Solution: Let A be the event that the student is from Hostel 7, let B be the event that the student is a Girl, and C be the event that the student's name starts with 'Z'. Finally, let D be the event that the students name starts with 'A'. Then $C \cap D$ is empty or equivalently $D \subset C^c$.

We are given $p(A) = 0.25$, $p(B) = 0.5$, $p(C) = 0.05$ and $p(A^c \cap B^c \cap D) = 0.7$. The laws of probability give us the inequalities:

$$\begin{aligned} p(A^c \cap B^c \cap D) &\leq p(B^c) \\ p(B^c) &= 1 - p(B) \end{aligned}$$

But this gives the contradition that $0.7 \leq 0.5$ which is impossible.

4. Let Ω the set of all people in the country and E be the subset of educated people. We can think of various collections S of people such as:

   - the collection of people with income less than M rupees a year.
   - the collection of people who own a cell phone.
   - the collection of people from Punjab.

   For each such S, let us define $E(S) = |S \cap E|/|E|$. Check that this satisfies the axioms of a probability function on the Boolean algebra of subsets of Ω. What is an interpretation of this measure as a probability in the naive sense?

Solution: $P(S)$ is the probability that a randomly chosen educated person will be in the collection S.

5. For each region R of the night sky, let $p(R)$ denote the percentage of visible objects that are seen in a photograph of that region. Moreover, suppose $q(R)$ is the percentage of visible galactic objects in R. Interpret $q(R)$ as a conditional probability. Suppose that 70% of all visible objects are in our galaxy. Write a formula for the (conditional) probability that an object from the region R is in our galaxy.

Solution: We can interpret $p(R)/100)$ as the probability $P(R)$ that a given visible object is in the region R. Then $q(R)/100$ is the conditional probability $P(R|G)$ that it is in region R given that it is in our galaxy. It is given that the probability $P(G)$ that an object is in our galaxy is $0.7 = 70/100$. From the formula for conditional probability $P(R|G)P(G) = P(R \cap G)$. Applying the formula again with R and G reversed, we see that $P(G|R) = P(R \cap G)/P(R) = 0.7 * q(R)/p(R)$.