Generalisations

Mathematician's like to generalise — this itself is a generalisation made by a mathematician!

In the earlier lecture we studied the probability of the event of r Heads arising out of k of flips of an unbiased idealised coin.

$$P(C_{r,k}) = \binom{k}{r}/2^k$$

What happens for a biased coin (for which distinct flips are still independent)? Let us say that the probability of a Head is now p (instead of 0.5). In that case the probability of a Tail is $1 - p$.

Arguing exactly as before one can show that the probability of an event like $H_1H_2T_3H_4T_5$ is $p^3(1-p)^2$ and so on. It follows that if we look for (the event) of a *given* sequence of Heads and Tails, the probability of this event *only* depends on the *number* of Heads and Tails; it is $p^r(1-p)k-r$ when there are r Heads and $k-r$ Tails in k flips.

Continuing the argument as before, the *number* of sequences that have r Heads and $k-r$ Tails is given by $\binom{k}{r}$. Since each of sequence is a *disjoint* event from every other sequence, we see that the probability of r Heads and $k-r$ Tails for a biased coin becomes:

$$P(C_{r,k}) = \binom{k}{r}p^r(1-p)^{k-r}$$

Dice

We can generalise this even further to multi-sided coins (also called dice).

- An s-sided die is associated with s distinct events $E_1, \ldots, E_s$. with $P(E_i) = p_i$ where $p_1 + \cdots + p_s = 1$.
- Multiple spins/flips of the die are independent of each other.

If we denote the event E_i for the k-th spin as $E_{i,k}$, then we have $P(E_{i,k}) = p_i$ and $E_{i,k}$ is independent of $E_{j_1,f_1} \cap \cdots \cap E_{j_l,f_l}$ as long as k is not one of the f_t's. (This notation is more complicated than the concept!).

Arguing as before we can see that the probability of a given sequence $(i_1, i_2, \ldots, i_k)$ as a result of k flips is given by

$$P(E_{i_1,1} \cap E_{i_2,2} \cap \cdots \cap E_{i_k,k}) = p_{i_1}p_{i_2}\cdots p_{i_k}$$

Let us now group these events according to "type". So suppose that a_1 of these events are of the type $E_{1,f}$ (for some f), a_2 of them are of the type $E_{2,f}$, and so on. We see that the probability of the event can be *re-written* as $p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s}$.

How many ways are there of making k choices of which a_1 are of type 1, a_2 are of type 2 and so on upto a_s of type s? The answer is given by the multinomial $\binom{k}{a_1,a_2,\ldots,a_s}$. (Note that $a_1 + \cdots + a_s = k$. By an argument similar to the one given earlier, we can provide the formula:

$$\binom{k}{a_1, \ldots, a_s} = \frac{k!}{a_1! \cdots a_s!}$$

As before, we note that it is not obvious that the right-hand side is an integer!

Using this, we see that counting the *number* of occurences of various sides of the die gives us the probability of the event $C_{(a_1,\ldots,a_s),k}$ where a die is rolled k types and shows the risft face a_1 times, the second face a_2 times and so on.

$$P(C_{(a_1,\ldots,a_s),k}) = \binom{k}{a_1, \ldots, a_s} p_1^{a_1} \ldots p_s^{a_s}$$

This is called the multinomial distribution.

Simulation and Limits

Mathematicians also try to unify things. So we can ask the question: Do we really need to have different dice? Can't we get sufficiently different probabilistic (stochastic) systems from coins?

We already saw that 3 coin flips result in 8 possible distinct/disjoint events which are equally likely. How can we reduce this to 6? The simple answer is to "throw away" two events!

Our new experiment is as follows.

1. We flip three coins.
2. If all three coins are different, we record the occurrence of events as follows: 1st event for the sequence (T, H, H), 2nd event for (H, T, H), 3rd event for (H, H, T), 4th event for (H, T, T), 5th event for (T, H, T) and 6th event for (T, T, H).
3. All three coins are the same, then we go back to step 1 and continue!

Intuitively, it is clear (!) that each of the events is equally likely. It follows that we have "constructed" a die out of coins. To provide a formal mathematical justification (in greater generality!) will take some more effort which we will see at a later date.