Flipping Coins

One of the simplest stochastic systems is the one associated with flipping a coin. While one could perhaps work out all the Physics associated with this process, the lack of knowledge of initial conditions and a number of instabilities in the associated system lead us to conclude that we cannot predict whether it will land "Head" or "Tail".

Hence, we define an "unbiased" coin as one where:

1. Each flip results in "Head" with probability 0.5.
2. Each flip results in "Tail" with probability 0.5.
3. Different multiple flips are independent.

It is physically possible for the coin to land on its edge and stay that way. However, we assign that event the probability of zero. One could also view 1-3 as simplifying assumptions which allow us to make calculations by treating an "idealized" coin.

Since we have assumed (3), we will see that it is possible to use the laws of probability to calculate probability of events involving multiple coin flips. Let H_n denote the event that we obtain a "Head" on the n-th coin flip and T_n denote the event that we obtain a "Tail" on the n-th coin flip. The above rules become:

1. $P(H_n) = 0.5$ for all n.
2. $P(T_n) = 0.5$ for all n.
3. $P(A_n|B_m) = P(A_n)$ when $n \neq m$, for each case of A and B from amongst H and T. More generally for any sequences N and M of flips that have no common flips the probability of a chosen sequence of Head and Tail events for each of them are independent.

We can now use the laws to calculate the probability of an event that gives a *precise* sequence of Head and Tail events for a large number of coin flips. For example,

$$P(H_1 \cap H_2) = P(H_1|H_2)P(H_2) = P(H_1)P(H_2) = 0.25$$

More generally, one can show that for each sequence of length k the probability of that particular sequence is $(1/2)^k$.

To improve our notation, let us (as above) number the coin flips as 1, 2, ..., k in sequence. For each subset I of $\{1, \ldots, k\}$ let H_I denote the event $\cap_{i \in I} H_i$ where the i-th flip results in Heads *if and only if* i is in I. By what we have written above $P(H_I) = 1/2^k$.

Each I represents a *distinct* sequence of heads and tails in the coin flip. So, if I and J are *distinct* (but not necessarily disjoint!), then $H_I \cap H_J =$. (Two events

A and B are said to be *mutually exclusive* if $P(A \cap B) = 0$.) We note that there are exactly 2^k choices for I since that is the number of subsets of $\{1, 2, \ldots, k\}$.

In summary, a single coin flip results in two equally likely mutually exclusive events called Head and Tail. This can be generalised to k coin flips which results in 2^k equally likely mutually exclusive events.

Counting and the Binomial Distribution

Coin flips can be "counted" or accumulated in a number of different ways. The simplest method involves counting the number of Heads that occur.

Thus, we can define the event $C_{r,k}$ as the event that there are (exactly) r Heads after k flips. Since all outcomes of a sequence of k flips are equally likely and have a probability of $1/2^k$. We only need to count the number of outcomes that result in exactly r heads. From the description above it is clear that is just the number of sets I in $\{1, 2, \ldots, k\}$ for which the cardinality $|I|$ is r. We define:

> $\binom{k}{r}$ is the number of ways of choosing exactly r elements out of a set of k elements.

We note that to get the algebraic expansion of $(a + b)^k$, we need to multiply out $(a + b) \cdot (a + b) \cdots (a + b)$ (k-times). In order to get the term $a^r b^{k-r}$ we would need to choose the a term in exactly r of the $(a + b)$'s and the b term in exactly $k - r$ of the $(a + b)$'s. This is what gives us the Binomial expansion:

$$(a+b)^k = \binom{k}{k} a^k + \binom{k}{k-1} a^{k-1} b + \cdots + \binom{k}{r} a^r b^{k-r} + \cdots + \binom{k}{0} b^k$$

For this reason, these numbers are also called binomial coefficients.

What is the "value" of $\binom{k}{r}$? One way to go about counting the possibilities is to pick the r elements one-by-one. There are k ways of pick the first element, $k - 1$ ways to pick the second element (since we need to leave out the element already picked), $k - 2$ ways to pick the third element (since we need to leave out the two elements already picked), and so on; there are $k - i + 1$ ways to pick the i-th element. This means that the number of ways to pick r elements *in* order* is $k \cdot (k - 1) \cdots (k - r + 1)$. We could have picked the same elements in a different order and this would give us the *same* subset! Thus we have over-counted each subset by the number of ways of re-ordering a set of order r; this is $r \cdot (r - 1) \cdots 1$, in other words $r!$ or factorial r. So we get the formula:

$$\binom{k}{r} = \frac{k \cdot (k-1) \cdots (k-r+1)}{r \cdot (r-1) \cdots 1}$$

Note that it is *not* obvious that the right-hand side is an integer. However, we *know* that this number counts something and so it is an integer!

There are a lot of different interesting identities associated with the binomial coefficients and we shall see some of them in this course.

Coming back to probability, we get the formula $P(C_{r,k}) = \binom{k}{r}/2^k$ as the probability of seeing r Heads in k coin flips. The function $f_k(r) = \binom{k}{r}/2^k$ is sometimes called the k-th Binomial distribution; we will later see why the term "distribution" is used in this context.

Walks and Games

A classical "random walk" is described as follows. A man starts at 0 on a straight line and moves left for Tail and right and right for Tail.

A more juicy description of a mathematically identical system is as follows.

A classical game is one played between two gamblers. We simplify the game as follows:

1. each gambler bets one rupee on the outcome of a coin flip by putting 1 rupee each in the "pot".
2. the coin is flipped and if it results in a Head, the first gambler
   (A) takes the pot. If it results in a Tail, the second gambler (B) takes the pot.

A different way to say this is to say that the gambler (A) wins 1 rupee for a Head and loses 1 rupee for Tail. When the game is paid k times, each of the above steps is carried out k times.

As usual, we mathematically idealise the situation and assume that the coin is not biased (even though we can't always be sure about this with gamblers!).

What is the probability that the gambler has won a rupees after k coin tosses? In order to do that there should be r Heads and $k-r$ Tails where $r-(k-r)=a$; in other words, we have $a = 2r-k$. Now, the probability can be calculated using the Binomial distribution. If $E_{a,k}$ denotes the event that the gambler wins a rupees in k flips (where we understand that if $a < 0$, this means that the gambler has *lost* $|a|$ rupees!), the probablity $P(E_{a,k}) = \binom{k}{(a+k)/2}/2^k$ when $(a+k)$ is divisible by 2.

So this is essentially the same as the binomial distribution except that the "center" has shifted.