

PHY 301: Classical Mechanics
Assignment Sheet 3: Central Forces and Calculus of Variations
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A. Central Forces

- 1. Geometry and terminology of a Kepler ellipse.** A particle of mass m moves in the attractive potential

$$V(r) = -\frac{k}{r}, \quad k > 0,$$

and its bound orbit is

$$\frac{1}{r} = \frac{mk}{\ell^2} (1 + \epsilon \cos \theta), \quad 0 \leq \epsilon < 1.$$

- (a) Write the orbit as $r = p/(1 + \epsilon \cos \theta)$ and identify p . What is p called geometrically?
- (b) Find the periapsis and apoapsis distances r_p and r_a .
- (c) Show that the semi-major axis is

$$a = \frac{p}{1 - \epsilon^2},$$

and hence that $r_p = a(1 - \epsilon)$ and $r_a = a(1 + \epsilon)$.

- (d) Find the semi-minor axis b in terms of p and ϵ .
- (e) If periapsis is on the positive x -axis and the force centre is at the origin, locate the centre and the second focus of the ellipse.
- (f) Show that

$$E = -\frac{k}{2a}, \quad \ell^2 = mka(1 - \epsilon^2).$$

- 2. Orbital speed and the vis-viva relation.** For an elliptic Kepler orbit of semi-major axis a and eccentricity ϵ :

- (a) Use angular-momentum conservation to obtain v_p/v_a , where v_p and v_a are the speeds at periapsis and apoapsis.
- (b) Show that

$$v_p^2 = \frac{k}{ma} \frac{1 + \epsilon}{1 - \epsilon}, \quad v_a^2 = \frac{k}{ma} \frac{1 - \epsilon}{1 + \epsilon}.$$

- (c) Show that at an arbitrary point of the orbit

$$v^2 = \frac{k}{m} \left(\frac{2}{r} - \frac{1}{a} \right)$$

(the vis-viva relation).

- (d) At periapsis the particle receives an instantaneous tangential increase Δv in speed. Find the condition on $v_p + \Delta v$ for the new orbit to become unbound.

- 3. Recovering the inverse-square force from the orbit.** Suppose an observed orbit about a force centre is

$$r(\theta) = \frac{p}{1 + \epsilon \cos \theta},$$

where p and ϵ are constants. Using Binet's equation, determine the central force $F(r)$. Show that the force must be inverse-square and find its strength in terms of p , m , and ℓ . Hence recover the relation between p and ℓ for the Kepler problem.

4. Runge–Lenz vector and orbit geometry. For $V(r) = -k/r$, define

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - mk\hat{\mathbf{r}}, \quad \mathbf{L} = \mathbf{r} \times \mathbf{p}.$$

- (a) Show directly that $\mathbf{A}$ is conserved.
- (b) Show that $\mathbf{A} \cdot \mathbf{L} = 0$.
- (c) Show that

$$\mathbf{A} \cdot \mathbf{r} = \ell^2 - mkr.$$

Taking the polar axis along $\mathbf{A}$, recover the conic orbit.

- (d) Show that

$$A = mk\epsilon.$$

Why does $\mathbf{A}$ point towards periapsis for an elliptic orbit?

- (e) Show that

$$A^2 = m^2k^2 + 2mE\ell^2,$$

and use this result to classify the orbit according to the sign of E .

5. Power-law forces, circular-orbit stability, and apsidal motion. A particle moves under the attractive central force

$$F(r) = -Cr^n, \quad C > 0.$$

- (a) Find the condition relating ℓ and the radius r_0 of a circular orbit.
- (b) Determine the range of n for which the circular orbit is stable.
- (c) For a small radial displacement from r_0 , find the radial oscillation frequency ω_r and compare it with the orbital angular frequency ω_θ .
- (d) Show that the angular separation between successive periapses of a nearly circular orbit is

$$\Delta\theta = \frac{2\pi}{\sqrt{n+3}}.$$

- (e) What do you obtain for the Kepler force and for the isotropic harmonic force?

6. Positive-energy Kepler motion in Levi–Civita coordinates. Introduce

$$x = u_1^2 - u_2^2, \quad y = 2u_1u_2, \quad r = u_1^2 + u_2^2,$$

and let $P_i = \partial L / \partial \dot{u}_i$ be the generalized momenta. For a Kepler trajectory with $E > 0$:

- (a) Show that

$$\frac{P_1^2 + P_2^2}{8m} - E(u_1^2 + u_2^2) = k.$$

How does this differ from the bound $E < 0$ case?

- (b) Define

$$B_{ij} = \frac{P_i P_j}{8m} - E u_i u_j.$$

Show that B_{ij} is conserved and that $\text{Tr } B = k$.

- (c) Show that

$$u_i B_{ij} u_j = kr - \frac{\ell^2}{2m}.$$

- (d) Choose fixed principal axes of B_{ij} and use the previous result to recover the physical conic orbit. Show that

$$\epsilon^2 = 1 + \frac{2E\ell^2}{mk^2} > 1.$$

Identify the orbit.

7. Kepler's third law from the auxiliary oscillator. For a bound Kepler orbit, let

$$D \equiv r \frac{d}{dt} = \frac{d}{d\tau}, \quad D^2 u_i + \Omega^2 u_i = 0, \quad \Omega^2 = \frac{|E|}{2m},$$

with

$$u_1 = \alpha \cos \Omega\tau, \quad u_2 = \beta \sin \Omega\tau, \quad dt = r d\tau.$$

(a) Show that one complete physical Kepler orbit corresponds to a change $\Delta(\Omega\tau) = \pi$, not 2π .

(b) Evaluate

$$T = \oint dt = \oint r d\tau$$

and express the answer in terms of α , β , and Ω .

(c) Using

$$a = \frac{\alpha^2 + \beta^2}{2}, \quad |E| = \frac{k}{2a},$$

show that

$$T = 2\pi \sqrt{\frac{ma^3}{k}}$$

and hence obtain Kepler's third law.

(d) Explain why the period is independent of the eccentricity.

8. Rotation of the Fradkin tensor and the Runge–Lenz vector. For a bound orbit define the conserved symmetric tensor in the u -plane

$$F_{ij} = \frac{P_i P_j}{8m} + |E| u_i u_j,$$

whose traceless part is related to the physical Runge–Lenz vector by

$$A_x = m(F_{22} - F_{11}), \quad A_y = -2mF_{12}.$$

Suppose the principal axes of F_{ij} are rotated through an angle ψ relative to the fixed u_1, u_2 axes, and the corresponding eigenvalues are F_1, F_2 .

(a) Express $F_{11} - F_{22}$ and $2F_{12}$ in terms of $F_1 - F_2$ and ψ .

(b) Show that the direction of the Runge–Lenz vector in the physical plane is rotated through 2ψ (up to the orientation set by the ordering of F_1, F_2).

(c) Explain the factor of two directly from

$$x + iy = (u_1 + iu_2)^2.$$

(d) Explain how the principal direction of F_{ij} determines the orientation of the major axis and the periapsis of the physical Kepler ellipse.

B. Calculus of Variations

9. Euler equation for a functional containing higher derivatives. Consider

$$J[y] = \int_{x_1}^{x_2} F(y, y', y'', y''', x) dx,$$

where y , y' , and y'' are fixed at both endpoints.

- (a) Starting from $y(x) \rightarrow y(x) + \varepsilon \eta(x)$, derive the Euler equation

$$\boxed{\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} + \frac{d^2}{dx^2} \frac{\partial F}{\partial y''} - \frac{d^3}{dx^3} \frac{\partial F}{\partial y'''} = 0.}$$

- (b) State the endpoint conditions on η , η' , and η'' .
 (c) How does the result simplify if F depends on y, y', y'' and x , but not on y''' ?

10. A sixth-order variational equation. Consider

$$J[y] = \frac{1}{2} \int_0^L [(y''')^2 + \lambda(y')^2] dx,$$

with y , y' , and y'' fixed at $x = 0, L$.

- (a) Derive the differential equation obeyed by a stationary function.
 (b) What is the order of the differential equation, and why is this consistent with the number of endpoint data specified?
 (c) Find the general form of the stationary function for $\lambda = 0$.

11. A first integral and a non-Euclidean geodesic. For $y(x) > 0$, consider

$$J[y] = \int_{x_1}^{x_2} \frac{\sqrt{1 + y'^2}}{y} dx.$$

- (a) Use the fact that the integrand has no explicit x dependence to obtain a first integral without solving the Euler equation directly.
 (b) Integrate the resulting first-order equation and show that the stationary curves are arcs of circles whose centres lie on the x -axis:

$$(x - x_0)^2 + y^2 = R^2.$$

- (c) Show that vertical straight lines $x = \text{constant}$ arise as the limiting case of this family when the circle radius and the magnitude of its centre coordinate tend to infinity together.

12. Variational problem with an integral constraint. Find the stationary functions of

$$J[y] = \int_0^L (y')^2 dx,$$

subject to

$$y(0) = y(L) = 0, \quad \int_0^L y^2 dx = C > 0.$$

- (a) Introduce a Lagrange multiplier and derive the Euler equation.
 (b) Show that nontrivial stationary functions exist only for a discrete set of multiplier values.
 (c) Determine the stationary functions satisfying the constraint.
 (d) Which stationary function gives the minimum value of J ?