

PHY301 Classical Mechanics

Problem Set IV

Velocity-dependent potentials, dissipation, non-holonomic constraints, and variational principles

This set is based on the topics covered in Part 3 of the lecture notes: velocity-dependent generalized potentials and electromagnetic coupling, Rayleigh dissipation, Lagrange equations with non-holonomic constraints, and the calculus of variations/Hamilton's principle. The problems are *not* direct repetitions of the worked derivations done in class.

Easy

Problem 1: Gauge transformation of the charged-particle Lagrangian

A particle of mass m and charge q moves in electromagnetic potentials $\phi(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$ with

$$L = \frac{1}{2}m\mathbf{v}^2 - q\phi + q\mathbf{A} \cdot \mathbf{v}.$$

Under a gauge transformation,

$$\mathbf{A}' = \mathbf{A} + \nabla\chi, \quad \phi' = \phi - \frac{\partial\chi}{\partial t}.$$

- (a) Find $L' - L$.
- (b) Show that $L' - L$ is a total time derivative.
- (c) Hence show that the equations of motion are unchanged.
- (d) Explain why adding a total time derivative to a Lagrangian does not change the dynamics.

Problem 2: Canonical versus mechanical momentum

For the charged particle of Problem 1,

$$L = \frac{1}{2}m\mathbf{v}^2 - q\phi + q\mathbf{A} \cdot \mathbf{v}.$$

- (a) Calculate the canonical momentum $\mathbf{p} = \partial L / \partial \mathbf{v}$.
- (b) Distinguish it from the mechanical momentum $m\mathbf{v}$.
- (c) Determine how $\mathbf{p}$ changes under the gauge transformation of Problem 1.
- (d) Show that the mechanical momentum is gauge independent.

Problem 3: Damped vertical oscillator using the Rayleigh function

A mass m hangs from a spring of spring constant k and experiences a viscous drag force $F_d = -c\dot{x}$. Take x positive downward from the natural length of the spring. Using the Rayleigh function convention

$$G = -\frac{1}{2}c\dot{x}^2,$$

- (a) Write T , U , and G .
- (b) Derive the equation of motion.
- (c) Shift the coordinate to the equilibrium position and simplify the equation.
- (d) State the conditions for underdamped, critically damped, and overdamped motion.

Problem 4: Torsional oscillator in a viscous medium

A disk of moment of inertia I is suspended by a torsion wire of torsional constant κ . The surrounding fluid exerts a viscous torque

$$\tau_d = -b\dot{\theta}.$$

- (a) Construct the Lagrangian and the Rayleigh dissipation function.
- (b) Derive the equation of motion.
- (c) For weak damping, find the damped angular frequency.
- (d) If the amplitude falls from A_0 to A_0/e in time τ , express b in terms of I and τ .

Problem 5: Is the constraint holonomic?

For each of the following constraints, determine whether it is holonomic or genuinely non-holonomic. Where possible, integrate it to an equivalent coordinate constraint.

- (a) $x dx + y dy = 0$, (b) $dy - x dx = 0$,
- (c) $\cos \theta dx + \sin \theta dy = 0$, (d) $dz - y dx = 0$.

Problem 6: A knife edge on a plane

A thin skate or knife edge moves on a horizontal plane. Let (x, y) be the position of its centre and let θ be the angle of the blade with the x -axis. The skate cannot move perpendicular to its blade.

- (a) Show that the velocity constraint is

$$-\sin \theta \dot{x} + \cos \theta \dot{y} = 0.$$

- (b) Explain why this is generally not equivalent to a single constraint $f(x, y, \theta) = 0$.
- (c) How many instantaneous degrees of freedom does the system possess?

Moderate

Problem 7: Damped pendulum with drag proportional to bob velocity

A pendulum of mass m and length ℓ moves in a vertical plane. The bob experiences viscous drag

$$\mathbf{F}_d = -c\mathbf{v}.$$

- (a) Find the Rayleigh function.
- (b) Express it entirely in terms of θ and $\dot{\theta}$.
- (c) Construct the Lagrangian.
- (d) Derive the exact nonlinear equation of motion.
- (e) Linearize about the stable equilibrium.
- (f) Determine the decay rate and damped frequency for small oscillations.

Problem 8: Two oscillators with dissipative coupling

Two equal masses m move on a horizontal frictionless line. Each is attached to a fixed wall by a spring of constant k , and the masses are coupled by a spring of constant k_c . A dashpot of damping coefficient c acts only between the two masses, so the dissipative force depends on their relative velocity.

- (a) Write T and U .
- (b) Show that an appropriate Rayleigh function is proportional to $(\dot{x}_1 - \dot{x}_2)^2$.
- (c) Derive the equations of motion.
- (d) Introduce normal coordinates and decouple the equations.
- (e) Which normal mode is damped? Is the other damped? Explain physically.

Problem 9: Charged particle in a uniform magnetic field: two gauges

A uniform magnetic field is given by

$$\mathbf{B} = B\hat{\mathbf{z}}.$$

Two vector potentials that produce this field are

$$\mathbf{A}_1 = (-By, 0, 0), \quad \mathbf{A}_2 = \frac{B}{2}(-y, x, 0).$$

Take $\phi = 0$.

- (a) Write the Lagrangian in each gauge.
- (b) Find $L_2 - L_1$.
- (c) Show that the difference is a total time derivative.
- (d) Derive the equations of motion.
- (e) Show that the trajectory in the xy -plane is a circle and obtain the cyclotron frequency.

Problem 10: Non-holonomic rolling disk on a horizontal plane

A vertical disk of radius R rolls without slipping on a horizontal plane. Let (x, y) be the coordinates of the point of contact projected onto the plane, θ the heading angle of the disk, and ϕ its spin angle.

- (a) Show that rolling without slipping implies

$$\dot{x} = R\dot{\phi} \cos \theta, \quad \dot{y} = R\dot{\phi} \sin \theta.$$

- (b) Write these constraints in Pfaffian form.
(c) Explain why these constraints do not globally eliminate coordinates.
(d) Write the unconstrained kinetic energy.
(e) Introduce Lagrange multipliers and write the constrained Lagrange equations.
(f) Interpret the multipliers physically.

Do not solve the full motion.

Problem 11: Sphere rolling on a horizontal plane

A homogeneous sphere of mass M , radius R , and moment of inertia I rolls without slipping on a horizontal plane.

- (a) Starting from

$$\mathbf{V}_{\text{CM}} + \boldsymbol{\omega} \times (-R\hat{\mathbf{z}}) = 0,$$

resolve the no-slip constraint into Cartesian component constraints.

- (b) Explain why these are velocity constraints.
(c) Introduce suitable generalized coordinates and write the kinetic energy.
(d) Set up, but do not solve, the Lagrange-multiplier equations.

Problem 12: Fermat's principle in a stratified medium

The refractive index depends only on height: $n = n(y)$. A light ray joining two fixed points makes the optical path stationary,

$$S[y] = \int n(y) \sqrt{1 + y'^2} dx.$$

- (a) Since the integrand has no explicit x -dependence, obtain a first integral.
(b) Show that it may be written as

$$n(y) \cos \alpha = \text{constant},$$

where α is the angle the ray makes with the horizontal.

- (c) Explain how this relates to Snell's law in a continuously varying medium.

Difficult

Problem 13: Soap film between two circular rings

Two coaxial circular rings of equal radius R are separated by distance $2d$. A soap film spanning them forms a surface of revolution. If the generating curve is $r = r(z)$, then the area is

$$A[r] = 2\pi \int_{-d}^d r \sqrt{1 + r'^2} \, dz.$$

- (a) Use the fact that the integrand has no explicit z -dependence to obtain a first integral.
- (b) Show that the extremal surface has the form

$$r(z) = r_0 \cosh\left(\frac{z}{r_0}\right).$$

- (c) Use the boundary condition $r(\pm d) = R$ to obtain the equation determining r_0 .
- (d) Explain qualitatively why a soap-film solution ceases to exist beyond a sufficiently large ring separation.

Problem 14: Hanging chain with fixed endpoints

A uniform flexible chain of linear mass density μ hangs under gravity between two fixed supports. Let its shape be $y(x)$.

- (a) Write the gravitational potential energy as a functional of $y(x)$, using $ds = \sqrt{1 + y'^2} \, dx$.
- (b) Impose the fixed-length condition by an undetermined multiplier.
- (c) Derive the Euler–Lagrange equation.
- (d) Show that the solution is a catenary,

$$y(x) = a \cosh\left(\frac{x - x_0}{a}\right) + C.$$

- (e) Give a physical interpretation of the multiplier.

Problem 15: Brachistochrone

A bead slides without friction under gravity from point A to a lower point B . Take y positive downward from A .

- (a) Using conservation of energy, show that the speed is

$$v = \sqrt{2gy}.$$

- (b) Show that the travel time is

$$T[y] = \int \sqrt{\frac{1 + y'^2}{2gy}} \, dx.$$

- (c) Use the fact that the integrand has no explicit x -dependence to obtain a first integral.
- (d) Show that the solution can be written parametrically as

$$x = a(\theta - \sin \theta), \quad y = a(1 - \cos \theta).$$

Problem 16: Geodesic on a cylinder

Two points lie on the surface of a circular cylinder of radius R . Using coordinates (ϕ, z) , the line element is

$$ds^2 = R^2 d\phi^2 + dz^2.$$

- (a) Write the length functional using $z = z(\phi)$.
- (b) Derive the Euler–Lagrange equation.
- (c) Show that

$$z(\phi) = A\phi + B.$$

- (d) Interpret the resulting curve in three dimensions.
- (e) Show explicitly that unwrapping the cylinder turns the geodesic into a straight line.

Problem 17: Action of the harmonic oscillator between fixed endpoints

For a one-dimensional harmonic oscillator,

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2,$$

with fixed endpoints

$$x(t_1) = x_1, \quad x(t_2) = x_2,$$

- (a) Use Hamilton’s principle to obtain the Euler–Lagrange equation.
- (b) Find the classical trajectory satisfying the endpoint conditions.
- (c) Evaluate the classical action

$$S_{\text{cl}} = \int_{t_1}^{t_2} L \, dt.$$

- (d) Discuss what happens when $\sin[\omega(t_2 - t_1)] = 0$.

Problem 18: Free-particle action and the nature of the extremum

A free particle moves from x_1 at t_1 to x_2 at t_2 with action

$$S[x] = \int_{t_1}^{t_2} \frac{1}{2}m\dot{x}^2 \, dt.$$

- (a) Find the stationary path.
- (b) Write

$$x(t) = x_{\text{cl}}(t) + \eta(t), \quad \eta(t_1) = \eta(t_2) = 0.$$

Expand the action to second order in η .

- (c) Show that

$$S[x] - S[x_{\text{cl}}] = \frac{m}{2} \int_{t_1}^{t_2} \dot{\eta}^2 \, dt.$$

- (d) Hence decide whether the stationary path is a minimum or merely stationary.

Problem 19: Several dependent variables: a particle in two dimensions

A particle moves in the plane in the potential

$$V(x, y) = \frac{k}{2}(x^2 + y^2) + \lambda xy.$$

The action is

$$S[x, y] = \int_{t_1}^{t_2} \left[\frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - V(x, y) \right] dt.$$

- (a) Vary $x(t)$ and $y(t)$ independently and obtain the Euler–Lagrange equations.
- (b) Transform to the coordinates

$$q_+ = \frac{x+y}{\sqrt{2}}, \quad q_- = \frac{x-y}{\sqrt{2}}.$$

- (c) Show that the action separates.
- (d) Find the two normal-mode frequencies.

Problem 20: Variational estimate with a trial path

A particle moves in the potential $V(x) = \frac{1}{2}kx^2$ from $x(0) = 0$ to $x(T) = X$. Instead of solving exactly, consider the one-parameter trial family

$$x_\alpha(t) = \frac{Xt}{T} + \alpha \sin\left(\frac{\pi t}{T}\right).$$

- (a) Verify that every member of the family satisfies the endpoint conditions.
- (b) Compute the action

$$S(\alpha) = \int_0^T \left[\frac{1}{2}m\dot{x}_\alpha^2 - \frac{1}{2}kx_\alpha^2 \right] dt.$$

- (c) Find the value of α for which $S(\alpha)$ is stationary.
- (d) Compare the resulting approximate path with the exact harmonic-oscillator trajectory in qualitative terms.
- (e) Explain why restricted trial functions can still yield useful variational approximations.

End of Problem Set IV