

Classical Mechanics: Problem Set II

Constraints, Virtual Work, D'Alembert's Principle and Lagrangian Mechanics

Instructions. The problems are grouped by difficulty. Show the essential steps in all derivations. Unless stated otherwise, constraints are ideal and surfaces are smooth. Use the sign conventions you state consistently.

Easy

Problem 1: Particle constrained to a parabola

A particle of mass m is constrained to move without friction on the curve

$$y = ax^2$$

in a vertical plane.

- Write the holonomic constraint.
- How many degrees of freedom does the particle have?
- Choose x as a generalized coordinate and write $\mathbf{r}(x)$.
- Find the velocity in terms of x and $\dot{x}$.
- Find the kinetic and potential energies.

Do not derive the equation of motion.

Problem 2: Bead on a translating circular wire

A bead moves without friction on a circular wire of radius a . The centre of the circle translates horizontally according to $X(t) = ut$. The circle remains vertical.

- Write the constraint in Cartesian coordinates.
- Is the constraint holonomic?
- Is it scleronomous or rheonomous?
- Choose an angular generalized coordinate θ and write $x(\theta, t)$ and $y(\theta, t)$.
- Find $\dot{x}$ and $\dot{y}$.

Problem 3: Generalized force on a bead on a straight wire

A bead is constrained to move along the straight line $y = \lambda x$ in the xy -plane. It is acted on by a constant force

$$\mathbf{F} = F_x \hat{\mathbf{x}} + F_y \hat{\mathbf{y}}.$$

Take x as the generalized coordinate.

- Write $\mathbf{r}(x)$.
- Calculate $\partial \mathbf{r} / \partial x$.
- Find the generalized force $Q_x = \mathbf{F} \cdot \partial \mathbf{r} / \partial x$.
- Explain why Q_x is generally not equal to F_x .

Problem 4: Virtual displacement on a sphere

A particle is constrained to move on the surface

$$x^2 + y^2 + z^2 = a^2.$$

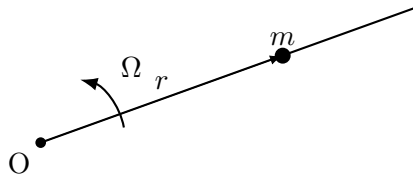
At a point (x, y, z) an infinitesimal virtual displacement is

$$\delta \mathbf{r} = \delta x \hat{\mathbf{x}} + \delta y \hat{\mathbf{y}} + \delta z \hat{\mathbf{z}}.$$

- (a) Derive the condition that $\delta x, \delta y, \delta z$ must satisfy.
- (b) Show geometrically that the admissible virtual displacement lies in the tangent plane to the sphere.
- (c) What is the direction of the constraint reaction for a smooth spherical surface?
- (d) Show that the constraint force does no virtual work.

Problem 5: Bead on a rotating straight rod

A smooth straight rod rotates in a horizontal plane about one end with constant angular velocity Ω . A bead of mass m can slide along the rod. Let r be the distance of the bead from the pivot.



- (a) Write the position of the bead in the inertial frame.
- (b) Calculate v^2 .
- (c) Obtain T .
- (d) There is no potential energy. Write the Lagrangian.
- (e) Derive the equation of motion for r .

Problem 6: Generalized force associated with an angle

A force $\mathbf{F}$ acts at the end of a massless rod of fixed length l pivoted at one end. Let θ be the angle of the rod in the plane.

- (a) Show that $Q_\theta = \mathbf{F} \cdot \partial \mathbf{r} / \partial \theta$.
- (b) Prove directly from the geometry that Q_θ equals the torque of $\mathbf{F}$ about the pivot.

Moderate**Problem 7: Two masses connected by a rigid rod on a smooth table**

Two particles m_1 and m_2 move on a horizontal frictionless plane and are connected by a rigid massless rod of length l .

- (a) Write the Cartesian constraint.

- (b) Show that admissible virtual displacements obey

$$(\mathbf{r}_1 - \mathbf{r}_2) \cdot (\delta \mathbf{r}_1 - \delta \mathbf{r}_2) = 0.$$

- (c) Show that the constraint forces on the two particles need not separately do zero virtual work.
 (d) Show that their total virtual work vanishes.

Problem 8: Mass sliding on the inside of a smooth cone

A particle of mass m is constrained to move on the inner surface of a smooth cone

$$z = \alpha r,$$

where (r, ϕ, z) are cylindrical coordinates.

- How many degrees of freedom are there?
- Choose suitable generalized coordinates.
- Express the kinetic energy entirely in terms of those coordinates.
- Write the gravitational potential energy.
- Construct the Lagrangian.
- Identify any cyclic coordinate and the associated conserved generalized momentum.

Problem 9: Particle on a smooth paraboloid

A particle moves under gravity on the surface

$$z = a(x^2 + y^2).$$

Use cylindrical coordinates (r, ϕ) .

- Write z in terms of r .
- Obtain T .
- Obtain U .
- Construct L .
- Show that ϕ is cyclic.
- Find the conserved momentum conjugate to ϕ .

Problem 10: Atwood machine using D'Alembert's principle

Two masses m_1 and m_2 are connected by a light inextensible string passing over a smooth massless pulley. Do *not* use Newton's equations for the two masses separately.

- Write the relation between the virtual displacements.
- Apply D'Alembert's principle directly.
- Derive the acceleration of the system.
- Explain explicitly how the tension disappears from the calculation.

Problem 11: Cylinder rolling without slipping down an incline

A rigid cylinder of mass M , radius R , and moment of inertia I about its symmetry axis rolls without slipping down an incline of angle α . Let x denote displacement of its centre along the incline and θ its rotation angle.

- (a) Write the rolling constraint.
- (b) Use it to reduce the system to one generalized coordinate.
- (c) Write the total kinetic energy.
- (d) Write the potential energy.
- (e) Derive the acceleration of the centre of mass using Lagrange's equation.
- (f) For a solid cylinder $I = \frac{1}{2}MR^2$, obtain a .

Problem 12: Bead on a rotating circular hoop

A bead of mass m slides without friction on a circular hoop of radius a . The hoop is vertical but rotates with constant angular velocity Ω about its vertical diameter. Let θ be measured from the downward vertical.

- (a) Write the coordinates of the bead in the laboratory frame.
- (b) Calculate the kinetic energy.
- (c) Write the potential energy.
- (d) Construct the Lagrangian.
- (e) Derive the equation of motion for θ .
- (f) Determine the equilibrium positions.

Problem 13: A mass attached to two springs

A particle of mass m moves along a horizontal line. It is attached to fixed walls by two springs with constants k_1 and k_2 . The springs are unstretched when $x = 0$.

- (a) Write T and U .
- (b) Obtain the equation of motion using Lagrange's equation.
- (c) Find the oscillation frequency.
- (d) Interpret the result in terms of an effective spring constant.

Difficult**Problem 14: Two coupled masses and three springs**

Two equal masses m move on a frictionless horizontal line. They are connected as

$$\text{wall} - k - m - k' - m - k - \text{wall}.$$

Let x_1, x_2 denote displacements from equilibrium.

- (a) Write T and U .
- (b) Obtain the two coupled Lagrange equations.
- (c) Assume $x_1 = Ae^{i\omega t}$, $x_2 = Be^{i\omega t}$.
- (d) Find the normal-mode frequencies.
- (e) Determine the corresponding ratios A/B .
- (f) Interpret the two modes physically.

Problem 15: Three identical masses on a ring

Three identical masses m lie at equally spaced positions on a circular ring and can move tangentially. Each neighbouring pair is connected by an identical spring of constant k . Let the small tangential displacements be q_1, q_2, q_3 .

- (a) Write the kinetic energy.
- (b) Write the potential energy.
- (c) Obtain the three coupled equations of motion.
- (d) Show that one normal mode has $\omega = 0$.
- (e) Explain physically what this mode represents.
- (f) Find the two nonzero normal-mode frequencies.

Problem 16: Bead on a rotating parabola

A bead of mass m is constrained to move on the curve

$$z = ar^2$$

which rotates about the vertical z -axis with constant angular velocity Ω . Use r as the generalized coordinate.

- (a) Write the position vector in the inertial frame.
- (b) Calculate the kinetic energy.
- (c) Calculate the potential energy.
- (d) Construct $L(r, \dot{r}, t)$.
- (e) Derive the radial equation.
- (f) Find the equilibrium radius r_0 .
- (g) Determine the condition under which a nonzero equilibrium radius exists.
- (h) Study small oscillations about r_0 .

Problem 17: Particle constrained to an ellipse

A particle of mass m moves under gravity on a smooth fixed ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

lying in a vertical plane. Use

$$x = a \sin \theta, \quad y = -b \cos \theta.$$

- (a) Find T and U .
- (b) Construct the Lagrangian.
- (c) Derive the exact equation of motion.
- (d) Find the stable equilibrium point.
- (e) Linearize the equation about that point.
- (f) Determine the frequency of small oscillations.
- (g) Show that the result reduces to that of a simple pendulum in the circular limit $a = b$.

Problem 18: Two particles constrained to perpendicular rods

Two equal masses m are constrained to move on two mutually perpendicular smooth rods along the x - and y -axes. Their positions are $(x, 0)$ and $(0, y)$. They are connected by a spring of natural length l_0 and spring constant k . Gravity is absent.

- (a) Find the instantaneous length of the spring.
- (b) Write the kinetic and potential energies.
- (c) Construct the Lagrangian.
- (d) Derive the coupled nonlinear equations of motion.
- (e) Find an equilibrium with $x = y = q_0$.
- (f) Linearize the equations about this equilibrium.
- (g) Find the two normal modes.

Problem 19: Bead on a freely rotating rod

A bead of mass m slides without friction on a massless rod. The rod rotates freely in a horizontal plane about one end. The bead position along the rod is r , and the rod orientation is θ . There is no external torque about the pivot.

- (a) Write the kinetic energy and the Lagrangian.
- (b) Show that θ is cyclic.
- (c) Obtain the conserved generalized momentum.
- (d) Use that conservation law to eliminate $\dot{\theta}$ from the radial equation.
- (e) Obtain an effective one-dimensional equation for r .
- (f) Explain physically why the angular velocity changes as the bead moves radially.

Problem 20: Sliding bead on a rotating spiral

A smooth wire in a horizontal plane has the form

$$r = a\theta.$$

The entire wire rotates about the origin at constant angular speed Ω . A bead of mass m moves along it. Let q denote the bead's angular coordinate relative to the wire, so that

$$\phi = q + \Omega t, \quad r = aq.$$

- (a) Write T entirely in terms of $q, \dot{q}, t$.
- (b) Construct the Lagrangian.
- (c) Derive the equation of motion.
- (d) Identify the terms that arise solely because the constraint itself is moving.
- (e) Determine whether the mechanical energy of the bead is conserved.
- (f) Explain your answer physically.

End of Problem Set II: 20 problems.