

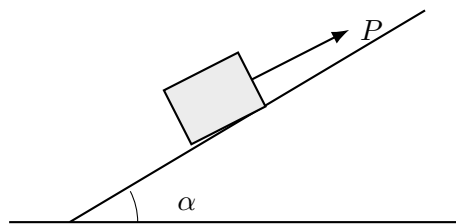
Classical Mechanics: Problem Set

Newtonian Mechanics, Coordinate Systems, Vectors and Tensors, Central Forces, and Rigid-Body Motion

Instructions. Show the essential steps in every derivation. For Problems 1–4, draw a complete free-body diagram before writing Newton's equations. Repeated Cartesian indices are summed unless stated otherwise. Problems 12–16 use a velocity field as an application of tensor methods; the emphasis is on transformations, rotations, and kinematics.

Problem 1: Block on a rough inclined plane

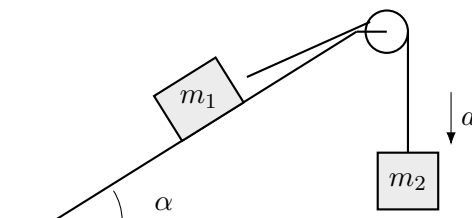
A block of mass m is pulled *up* a plane inclined at an angle α to the horizontal by a force P parallel to the plane. The coefficients of static and kinetic friction are μ_s and μ_k , respectively.



- Draw the free-body diagram of the block, showing the weight, normal reaction, applied force and friction.
- Find the minimum value of P required to start moving the block upward.
- If P is larger than this value and the block is moving upward, find its acceleration.
- What range of P allows the block to remain at rest?

Problem 2: Two masses connected by a string

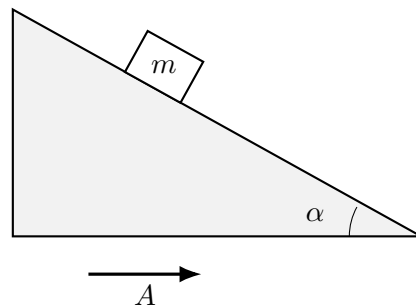
A block of mass m_1 rests on a rough plane of angle α and is connected by a light inextensible string over a smooth, massless pulley to a hanging mass m_2 . The coefficient of kinetic friction between m_1 and the plane is μ_k . Assume that m_2 is large enough that the motion is as shown: m_2 moves downward and m_1 moves up the plane.



- Draw separate free-body diagrams for m_1 and m_2 .
- Find the acceleration a of the system.
- Find the string tension T .
- State the inequality that must be satisfied for the assumed direction of motion to be self-consistent.

Problem 3: Block at rest relative to an accelerating wedge

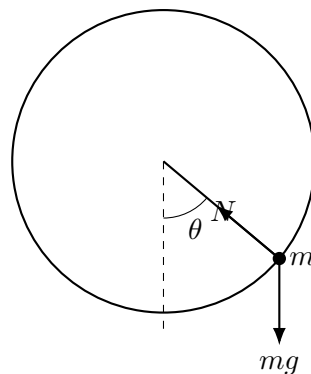
A smooth wedge of angle α accelerates to the right with constant acceleration A . A small block of mass m rests on the smooth inclined face. Find the value of A for which the block remains at a fixed point on the wedge.



- In the inertial ground frame, draw the free-body diagram and resolve Newton's equation into horizontal and vertical components.
- Find A and the normal reaction N .
- Re-derive A in the non-inertial frame of the wedge by introducing the appropriate inertial (pseudo) force.

Problem 4: Particle on the inside of a vertical circular track

A particle of mass m moves without friction on the inside of a vertical circular track of radius R . At a point where the radius to the particle makes an angle θ with the vertically downward direction, its speed is v .



- Draw the free-body diagram and write the radial equation of motion.
- Find the normal reaction N in terms of m, R, v, θ .
- If the particle completes a vertical circle while remaining in contact with the track, what is the minimum possible speed at the top of the circle?
- If it is projected from the bottom, use energy conservation to find the minimum speed at the bottom needed to complete the circle while maintaining contact.

Problem 5: Is it a vector?

In two dimensions define

$$A_1 = x^2 - y^2, \quad A_2 = 2xy.$$

Under a passive rotation of the coordinate axes through θ ,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

- (a) Find A'_1 and A'_2 in terms of A_1, A_2 .
- (b) Does (A_1, A_2) transform as an ordinary vector? Explain.
- (c) What rotation angle does the pair (A_1, A_2) effectively experience?

Problem 6: Is it really a vector?

Consider in two dimensions the quantities

$$A_1 = x^2 - y^2, \quad A_2 = 2xy.$$

Under a rotation through an angle θ ,

$$x'_i = R_{ij}x_j.$$

- (a) Find A'_1 and A'_2 .
- (b) Determine whether A_i transforms as a vector.
- (c) For a rotation through $\theta = \pi/4$, compare the transformed pair with what an ordinary vector would do.
- (d) Explain why specifying “magnitude and direction” is not by itself a sufficient mathematical definition of a vector.

Problem 7: Polar vector versus axial vector

Let

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}.$$

Consider the reflection

$$x' = -x, \quad y' = y, \quad z' = z.$$

- (a) Find the transformation matrix R and its determinant.
- (b) Determine how $\mathbf{r}$ and $\mathbf{p}$ transform.
- (c) Calculate $\mathbf{L}'$ directly from $\mathbf{r}' \times \mathbf{p}'$.
- (d) Show that

$$L'_i = (\det R) R_{ij} L_j.$$

- (e) Explain the distinction between a polar vector and an axial vector.

Problem 8: Basic Levi–Civita identities

Starting from

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl},$$

derive

$$\epsilon_{ijk}\epsilon_{ijl} = 2\delta_{kl}$$

and

$$\epsilon_{ijk}\epsilon_{ijk} = 6.$$

Do not evaluate the sums by explicitly listing all components of ϵ_{ijk} .

Problem 9: Vector identities using ϵ_{ijk}

Using index notation and the Levi-Civita symbol, prove

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}).$$

Then prove

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}).$$

Problem 10: Why angular momentum is an axial vector

Write angular momentum as

$$L_i = \epsilon_{ijk} r_j p_k.$$

Under a general orthogonal transformation,

$$r'_i = R_{ij} r_j, \quad p'_i = R_{ij} p_j.$$

Using

$$\epsilon_{ijk} R_{jl} R_{km} = (\det R) R_{in} \epsilon_{nlm},$$

show that

$$L'_i = (\det R) R_{ij} L_j.$$

Hence explain why angular momentum behaves like an ordinary vector under proper rotations but differently under reflections.

Problem 11: Antisymmetric tensors and axial vectors

Let A_{ij} be a three-dimensional antisymmetric tensor,

$$A_{ij} = -A_{ji}.$$

Define

$$a_i = -\frac{1}{2} \epsilon_{ijk} A_{jk}.$$

(a) Show that

$$A_{ij} = -\epsilon_{ijk} a_k.$$

(b) How many independent components does A_{ij} possess?

(c) Show that the independent components can therefore be represented by the three-component quantity a_i .

(d) Determine whether a_i is a polar or an axial vector.

Problem 12: Velocity gradient—deformation versus rotation

Consider a velocity field $\mathbf{v}(\mathbf{r})$ and define

$$L_{ij} \equiv \frac{\partial v_i}{\partial x_j}.$$

Decompose it as

$$L_{ij} = D_{ij} + W_{ij},$$

where

$$D_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right), \quad W_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right).$$

- (a) Show that $D_{ij} = D_{ji}$ and $W_{ij} = -W_{ji}$.
- (b) Under a rotation $x'_i = R_{ij}x_j$, $v'_i = R_{ij}v_j$, show that

$$L'_{ij} = R_{ik}R_{jl}L_{kl}.$$

- (c) Hence show separately that

$$D' = RDRT^T, \quad W' = RWRT^T.$$

- (d) Are the individual components of D_{ij} and W_{ij} rotationally invariant?
- (e) Are D_{ij} and W_{ij} nevertheless tensors?
- (f) Explain carefully the difference between invariance and covariance under rotations.

Problem 13: The antisymmetric part as local rigid-body rotation

For a rigid body rotating with constant angular velocity $\boldsymbol{\Omega}$,

$$\mathbf{v} = \boldsymbol{\Omega} \times \mathbf{r}, \quad v_i = \epsilon_{ijk}\Omega_j x_k.$$

- (a) Calculate $\partial v_i / \partial x_j$.
- (b) Show that $D_{ij} = 0$.
- (c) Show that, with the convention in Problem 12,

$$W_{ij} = -\epsilon_{ijk}\Omega_k.$$

- (d) Explain physically why a rigidly rotating material element has a nonzero velocity gradient but experiences no deformation.

Problem 14: Vorticity from the antisymmetric tensor

Define the vorticity by

$$\boldsymbol{\omega} = \nabla \times \mathbf{v}, \quad \omega_i = \epsilon_{ijk} \frac{\partial v_k}{\partial x_j}.$$

- (a) Show that

$$W_{ij} = -\frac{1}{2}\epsilon_{ijk}\omega_k.$$

- (b) For rigid rotation $\mathbf{v} = \boldsymbol{\Omega} \times \mathbf{r}$, show that

$$\boldsymbol{\omega} = 2\boldsymbol{\Omega}.$$

- (c) Explain why vorticity is an axial rather than a polar vector.

Problem 15: Why viscous stress contains the symmetric part

For an incompressible Newtonian fluid, the viscous stress is

$$\tau_{ij} = 2\eta D_{ij}.$$

Consider the two flows

$$(A) \quad v_x = -\Omega y, \quad v_y = \Omega x,$$

and

$$(B) \quad v_x = \dot{\gamma} y, \quad v_y = 0.$$

For each flow:

- (a) calculate L_{ij} ,
- (b) calculate D_{ij} ,
- (c) calculate W_{ij} ,
- (d) calculate τ_{ij} ,
- (e) identify whether the motion contains deformation or rigid rotation.

Finally explain why a proposed relation $\tau_{ij} \propto \partial v_i / \partial x_j$ would give an unphysical result for rigid-body rotation.

Problem 16: Infinitesimal separation of neighbouring material points

Two neighbouring material points have separation dx_i . Their relative velocity is

$$dv_i = L_{ij} dx_j.$$

Show that

$$\frac{d}{dt}(d\ell^2) = 2D_{ij} dx_i dx_j, \quad d\ell^2 = dx_i dx_i.$$

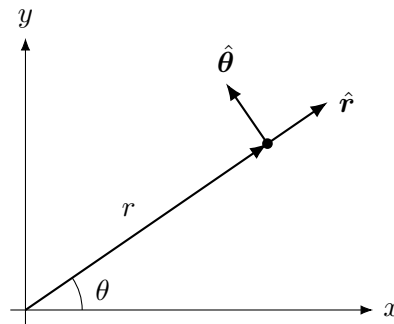
Hence show explicitly that the antisymmetric part W_{ij} makes no contribution to the instantaneous change in distance between neighbouring points. What physical property of W_{ij} does this demonstrate?

Problem 17: Plane-polar-coordinate kinematics and a central force

In plane polar coordinates,

$$\mathbf{r} = r\hat{\mathbf{r}}.$$

The geometry is shown below.



Starting from

$$\dot{\hat{\mathbf{r}}} = \dot{\theta}\hat{\boldsymbol{\theta}}, \quad \dot{\hat{\boldsymbol{\theta}}} = -\dot{\theta}\hat{\mathbf{r}},$$

derive

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$$

and

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}.$$

For a central force $F(r)\hat{\mathbf{r}}$, show that

$$r^2\dot{\theta} = \text{constant}$$

and interpret the constant in terms of angular momentum.

Problem 18: Spherical-polar-coordinate acceleration and central forces

Using

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} + r\sin\theta\dot{\phi}\hat{\boldsymbol{\phi}},$$

derive

$$\begin{aligned} a_r &= \ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2, \\ a_\theta &= r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\dot{\phi}^2, \\ a_\phi &= r\sin\theta\ddot{\phi} + 2\dot{r}\sin\theta\dot{\phi} + 2r\cos\theta\dot{\theta}\dot{\phi}. \end{aligned}$$

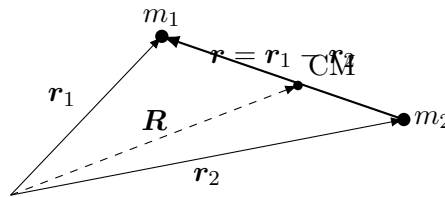
For a central force:

- use $a_\phi = 0$ to obtain a conserved quantity;
- relate it to a component of angular momentum;
- explain why it is always possible to choose a coordinate system in which the motion occurs in the plane $\theta = \pi/2$.

Problem 19: Reduction of the two-body problem

Two particles of masses m_1 and m_2 interact through a central potential $V(|\mathbf{r}_1 - \mathbf{r}_2|)$. Define

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}, \quad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2.$$



- Obtain $\mathbf{r}_1 - \mathbf{R}$ and $\mathbf{r}_2 - \mathbf{R}$ in terms of $\mathbf{r}$.
- Show that

$$T = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2, \quad M = m_1 + m_2, \quad \mu = \frac{m_1m_2}{m_1 + m_2}.$$

- Show that the relative-coordinate equation is

$$\mu\ddot{\mathbf{r}} = -\nabla V(r).$$

- Explain the limit $m_2 \gg m_1$.

Problem 20: Deriving the moment-of-inertia tensor using Levi-Civita symbols

For particles undergoing rigid-body rotation,

$$\mathbf{v}_\alpha = \boldsymbol{\omega} \times \mathbf{r}_\alpha.$$

Start from

$$L_i = \sum_{\alpha} m_{\alpha} \epsilon_{ijk} r_{\alpha j} v_{\alpha k}.$$

Substitute

$$v_{\alpha k} = \epsilon_{klm} \omega_l r_{\alpha m}$$

and use a contraction identity for two Levi-Civita symbols to show that

$$\mathbf{L} = I_{ij} \omega_j,$$

where

$$I_{ij} = \sum_{\alpha} m_{\alpha} (r_{\alpha}^2 \delta_{ij} - r_{\alpha i} r_{\alpha j}).$$

Problem 21: Rotational kinetic energy and the inertia tensor

Using

$$I_{ij} = \sum_{\alpha} m_{\alpha} (r_{\alpha}^2 \delta_{ij} - r_{\alpha i} r_{\alpha j}),$$

show that

$$T_{\text{rot}} = \frac{1}{2} I_{ij} \omega_i \omega_j.$$

Then prove that I_{ij} is symmetric. Explain why symmetry guarantees that its eigenvalues are real and that eigenvectors belonging to distinct eigenvalues can be chosen orthogonal. Under what condition are $\mathbf{L}$ and $\boldsymbol{\omega}$ parallel?

Problem 22: Principal axes

A rigid body has inertia tensor

$$I = I_0 \begin{pmatrix} 5 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

- Find the three principal moments.
- Find the corresponding normalized principal axes.
- Construct an orthogonal matrix R that transforms into the principal-axis frame.
- Verify explicitly that $I' = R I R^T$ is diagonal.
- If $\boldsymbol{\omega} = \omega_0(1, 0, 0)$ in the original coordinates, determine $\mathbf{L}$.
- Are $\mathbf{L}$ and $\boldsymbol{\omega}$ parallel?

Problem 23: Rotation and deformation in a general linear flow

Consider the linear velocity field

$$\mathbf{v} = \begin{pmatrix} a & -\Omega + \gamma & 0 \\ \Omega + \gamma & -a & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{r}.$$

- Calculate L_{ij} .
- Decompose it into D_{ij} and W_{ij} .
- Determine the vorticity.

- (d) Find the eigenvalues and eigenvectors of D_{ij} .
- (e) What directions experience maximum extension and compression?
- (f) Under what conditions does the flow reduce to pure rigid rotation?
- (g) Under what conditions is it pure strain (i.e. $W_{ij} = 0$)?
- (h) Verify that $D_{ij}D_{ij}$ and $W_{ij}W_{ij}$ are unchanged by an arbitrary rotation of coordinates.

End of problem set: 23 problems.