Brief Solutions to End Semester Examination

The following are the key ideas to the solutions to questions posed in the End Semester examination.

Question 1

Consider the small category $\mathcal{C}$ with objects as the sets $[0, n] = \{0, 1, \ldots, n\}$ for natural numbers $n \in \mathbb{N}$ and morphisms as *injective* set-maps $[0, n] \to [0, m]$. Consider the poset $\mathbb{N}$ (with the usual order) as a category.

a. Is there a functor $\mathcal{C}$ to $\mathbb{N}$ which is one-to-one on objects? If so, describe it.
b. Is there a functor $\mathbb{N}$ to $\mathcal{C}$ which is one-to-one on objects? If so, describe it.

Solution 1

a. There is an injective set-map $f : [0, n] \to [0, m]$ if and only if $n \leq m$. Thus, the functor is identity on objects and sends every set-map f as above to the *unique* map $n \to m$ (since $n \leq m$) in the poset $\mathbb{N}$ considered as a category.

b. Given $n \leq m$, there is a natural injective map $i_{n,m} : [0, n] \to [0, m]$ which sends each k in the domain to k in the range. It is clear that this is preserved under composition. Thus, we have a functor which is identity on objects and sends $n \to m$ in $\mathbb{N}$ to the set-map $i_{n,m}$.

Question 2

The category $\mathcal{C}$ consists of two objects 0 and 1, and has only one non-identity morphism $a : 0 \to 1$.

Decribe the category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ of functors from $\mathcal{C}$ to $\mathcal{D}$ in terms of objects and morphisms of the category $\mathcal{D}$.

Solution 2

Since a functor takes identity morphisms to identity identity morphisms, a functor F from $\mathcal{C}$ to $\mathcal{D}$ is fully described once we know $F(a) : F(0) \to F(1)$, which is a morphism in $\mathcal{D}$ between the objects $F(0)$ and $F(1)$. Thus, objects of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ can be identified with morphisms in $\mathcal{D}$.

A morphism in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is a natural transformation $\eta : F \to G$ of functors. This means that we have morphisms $\eta(0) : F(0) \to G(0)$ and $\eta(1) : F(1) \to G(1)$ such that $\eta(1) \circ F(a) = G(a) \circ \eta(0)$. In other words, we have a commutative diagram:

$$\begin{array}{ccc} F(0) & \xrightarrow{\eta(0)} & G(0) \\ \downarrow{\scriptstyle F(a)} & & \downarrow{\scriptstyle G(a)} \\ F(1) & \xrightarrow{\eta(1)} & G(1) \end{array}$$

Question 3

Given a category $\mathcal{C}$, is there are category $\mathcal{C}_e$ with the same objects as $\mathcal{C}$ such that every morphism in $\mathcal{C}_e$ is an epic morphism? If so, describe this category in terms of objects and morphisms of $\mathcal{C}$.

Solution 3

One can give a "trivial" solution by having *no* non-identity morphisms in $\mathcal{C}_e$. Alternatively, one can say that *every* morphism in $\mathcal{C}_e$ is an isomorphism.

> So the question should really have put the requirement that $\mathcal{C}_e$ be a subcategory of $\mathcal{C}$!

The main point to note is that (as proved in the notes) the composition of epic morphisms is an epic morphism and identity is also an epic morphism.

Thus, we can take the category $\mathcal{C}_e$ to have the same objects as $\mathcal{C}$. We define $\mathcal{C}_e(A, B)$ to be those elements of $\mathcal{C}(A, B)$ which are epic. By the previous remark, this is closed under composition and contains identity.

One final thing that needs to be remarked is that every morphism in $\mathcal{C}_e$ is epic *as a morphism in* $\mathcal{C}_e$. This follows since the condition f to be epic is that if $g \circ f = h \circ f$ then $g = h$. Since f is given, this is true for g, h morphisms in $\mathcal{C}$; so it is *a fortiori* true since $\mathcal{C}_e(A, B)$ is a subset of $\mathcal{C}(A, B)$.

Question 4

Fix a group G. Consider the functor M from **Set** to itself which sends a set T to the set $G \times T$ and a set-map $f : T \to S$ to the set-map $(1_G, f) : G \times T \to G \times S$. Define natural transformations

$$\begin{aligned} u_T : T \to MT = G \times T \qquad & \text{by } u_T(t) = (e, t) \\ v_T : MMT = G \times G \times T \to MT = G \times T \qquad & \text{by } v_T(g, h, t) = (g \cdot h, t) \end{aligned}$$

Where $e \in G$ is the identity element and $g \cdot h$ is the product of elements g and h in G.

a. Show that (M, u, v) is a monad.
b. Describe in terms of set theory what an M-algebra is.

Solution 4

a. If you look at the solution to Quiz 10, then you can see that $M = FL$ where F and L are the pair of adjoint functors defined in that solution! Moreover, u is the unit of that adjunction and v is derived from the co-unit of that adjunction in the usual way. That can be used to justify that (M, u, v) is a monad. One can also check the axioms for a monad directly by elementary group theory.

b. An M-algebra is a set T with a morphism $MT = G \times T \to T$ which satisfies various commutative diagrams. One checks that these diagrams imply that this map gives action of G on T. Thus M-algebras are the same as G-sets.

Question 5

Consider the category $\mathcal{A}$ defined as follows:

- Objects are pairs (S, b) where S is a set and $b : S \to S$ is a bijection.
- A morphism $f : (S, b) \to (T, c)$ is a set-map $f : S \to T$ such that $f \circ b = c \circ f$.

Consider the forgetful functor F from $\mathcal{A}$ to **Set** that associates to (S, b) the set S and to the morphism $f : (S, b) \to (T, c)$, the set-map $f : S \to T$.

Does this functor have a left-adjoint? If so, what is it?

Solution 5

This is *also* an application of the solution to Quiz 11. A set S with a bijection $b : S \to S$ is the same as a $\mathbb{Z}$ action on S. Moreover, morphisms $\mathcal{A}$ can be easily checked to be morphisms of $\mathbb{Z}$-sets. Thus, we can apply the solution of Quiz 11 to conclude that the left adjoint of this functor associates to S, the set $\mathbb{Z} \times S$ with the bijection $(n, s) \mapsto (n + 1, s)$.

The various identifications required to verify that this is a left-adjoint are easily checked.