Quiz 11: Products over posets

Question

Let P be the category (poset) whose objects are positive rational numbers with a (unique) morphism $r \to s$ if $\frac{s}{r} \in \mathbb{N}$.

Given D from P to **Ab** (the category of abelian goups) which sends r to the subgroup

$$D(r) = \{q \in \mathbb{Q} : rq \in \mathbb{Z}\} \subset \mathbb{Q}$$

Note that $\frac{r}{s} \in \mathbb{N}$ means that $D(r) \subset D(s)$ in a natural way.

1. Does the product $\prod_P D$ exist? If yes, what is it?
2. Does the co-product $\coprod_P D$ exist? If yes, what is it?

Answer

We note that there is a natural inclusion $D(r) \subset \mathbb{Q}$ for each r in P. Moreover, this is compatible with the inclusion $D(r) \subset D(s)$ associated with the morphism $r \to s$ in P. This shows that if the co-product exists, then it has a natural morphism to $\mathbb{Q}$. We claim that $\mathbb{Q}$ *is* the co-product in **Ab**.

Given an abelian group A and homomorphisms $a_r : D(r) \to A$ that are compatible with the inclusions $D(r) \subset D(s)$ when $r \to s$. Given p/q in $\mathbb{Q}$ such that p and q have no common factor. We see that p/q lies in $D(q)$. So we can try to define $a(p/q) = a_r(p/q)$ provided we check that this is compatible for the inclusion $D(r) \subset D(s)$ when $r \to s$. We are *given* that a_s restricts to a_r on elements of $D(r)$. Thus $a_s(p/q) = a_r(p/q)$. This shows that $a : \mathbb{Q} \to A$ is well-defined.

Thus, the co-product is $\mathbb{Q}$.

To calculate the product, suppose we are given an abelian group A and homomorphisms $d_r : A \to D(r)$ which are compatible with the inclusions $D(r) \subset D(s)$. Given any non-zero element p/q in $D(s)$, we can find an r such that p/q is *not* in $D(r)$. This means that p/q cannot be in the image of d_s and hence it cannot be in the image of d_r due to compatibility. This shows that $d_r(A) = \{0\}$.

Thus, the product is $\{0\}$.