Quiz 9: Adjoint Functor Theorem for Posets

Question

Given $\mathcal{U}$ is the poset of open subsets of the usual topological space $\mathbb{R}$ of real numbers abd $\mathcal{P}$ is the poset of *all* subsets of $\mathbb{R}$. Consider these posets as categories and let $F : \mathcal{U} \to \mathcal{P}$ denote the usual inclusion considered as a functor.

1. Does F have a left adjoint?
2. Does F have a right adjoint?
3. Explain your answer with reference to the "Adjoint functor theorem for posets".

Answer

As explained in the notes, a product in a poset P considered as a category is the greatest lower bound of a subset $S \subset P$. Similarly, a co-product is the least upper bound of a subset $S \subset P$.

In the poset $\mathcal{P}$ which is the power set of a set R, the order relation is the inclusion of sets in other sets. It follows that, given a subset $\mathcal{S}$ of $\mathcal{P}$, its least upper bound is precisely the union $\cup_{A \in \mathcal{S}} A$ and its greatest lower bound is precisely the intersection $\cap_{A \in \mathcal{S}} A$.

Now, the union of a collection of open sets is in a topological space is *also* an open set. On the other hand the intersection of a collection of open sets is not, in general an open set. The *largest* open set contained in the intersection $\cap_{A \in \mathcal{S}}$, where $\mathcal{S}$ is a collection of open sets, is the *interior* of this intersection $\text{int}\,(\cap_{A \in \mathcal{S}})$.

It follows that the functor F *preserves* co-products, but does *not*, in general, preserve products. Thus, we can expect it to *be* the left-adjoint of a right-adjoint functor G. However, it cannot be the right-adjoint of a left-adjoint functor H.

In fact, we have a functor $G : \mathcal{P} \to \mathcal{U}$ which takes a set A to its interior $\text{int}(A)$. Clearly this functor satisfies the following:

> Given a set A and an open set U, we have $U \subset A$ if and only if $U \subset int(A)$.

This shows that G is the right-adjoint to F (which takes U to U).

On the other hand, if we had a left-adjoint functor H to F, then would need to satisfy:

> Given a set A and an open set U, we have $A \subset U$ if and only if $A \subset H(U)$.

In other words, we would need to find a "smallest" open set containing a given set. When we take $A = [-1, 1]$ we see that there is no such smallest open set. Any open set U containing A contains $U_n = (-1 - 1/n, 1 + 1/n)$ for a large enough and the intersection of U_n's is precisely $A = [-1, 1]$. In fact, this also gives an example where the product in $\mathcal{U}$ is $(-1, 1)$ which is strictly smaller than the product $[-1, 1]$ in $\mathcal{P}$. Thus, F does not preserve products.