Quiz 7: Products

Question

Consider the category $\mathcal{C}$ whose objects are finite cyclic groups and morphisms are surjective group homomorphisms. Find the product (if it exists) of $\mathbb{Z}/\langle 4\rangle$ and $\mathbb{Z}/\langle 6\rangle >$.

Answer

To get a surjective group homomorphism $\mathbb{Z}/\langle n\rangle \to \mathbb{Z}/\langle m\rangle$, we need the element 1 on the left-hand side to go to a generator on the right hand side. This means that $m \cdot 1$ should be 0 on the left-hand side. In turn, this means that n divides m.

Conversely, if n divides m, then we can *define* a surjective group homomorphism $\mathbb{Z}/\langle n\rangle \to \mathbb{Z}/\langle m\rangle$ by sending 1 on the left-hand side to 1 on the right-hand side.

Thus, to get surjective group homomorphisms $\mathbb{Z}/\langle n\rangle \to \mathbb{Z}/\langle 4\rangle$ and $\mathbb{Z}/\langle n\rangle \to \mathbb{Z}/\langle 6\rangle$, we need n to be a multiple of 4 and 6. In other words, n is a multiple of the *least common multiple* of 4 and 6 which is 12.

As seen above, there *are* surjective group homomorphisms $\mathbb{Z}/\langle 12\rangle \to \mathbb{Z}/\langle 4\rangle$ and $\mathbb{Z}/\langle 12\rangle \to \mathbb{Z}/\langle 6\rangle$.

Moreover, given surjective group homomorphisms $\mathbb{Z}/\langle n\rangle \to \mathbb{Z}/\langle 4\rangle$ and $\mathbb{Z}/\langle n\rangle \to \mathbb{Z}/\langle 6\rangle$, we see that n must be a multiple of 12.

To prove that $\mathbb{Z}/\langle 12\rangle$ *is* the product, we need to show that given that the first map sends 1 to a and the second map sends 1 to b, there is a *unique* element c in $\mathbb{Z}/\langle 12\rangle$ such that sending 1 to c gives the required surjective map $\mathbb{Z}/\langle 12\rangle$.

In other words, given a generator a of $\mathbb{Z}/\langle 4\rangle$ and a generator b of $\mathbb{Z}/\langle 6\rangle$, there is a unique choice c of generator of $\mathbb{Z}/\langle 12\rangle$ which maps to a and b under the maps $\mathbb{Z}/\langle 12\rangle \to \mathbb{Z}/\langle 4\rangle$ and $\mathbb{Z}/\langle 12\rangle \to \mathbb{Z}/\langle 6\rangle$ that send 1 to 1. This means that we need to solve

$$\begin{aligned} c &= a + 4r \pmod{12} \\ c &= b + 6s \pmod{12} \end{aligned}$$

given that $\gcd(a, 4) = 1 = \gcd(b, 6)$ and we need to show that the solution is unique as an element of $\mathbb{Z}/\langle 12\rangle$. Suppose that $(12/4)p+(12/6)q = 1$ (for example $p = 1$ and $q = -1$). Then we easily check that $c = (12/4)ap + (12/6)bq$ is the unique solution to these equations modulo 12. We can also use the “Chinese Remainder Theorem”.

Note 1 Note that the above method generalises by replacing 4 by M and 6 by N if we replace 12 by the least common multiple of M and N.

Note 2 If we "expand" the category $\mathcal{C}$ to the $\mathcal{D}$ which contains *all* homomorphisms between cyclic groups (objects of $\mathcal{D}$ are still only cyclic groups), then the product does not exist in $\mathcal{D}$! This can be seen as follows.

1. If $\alpha \circ \beta$ is a surjective homomorphism of groups then α is surjective.
2. We have surjective group homomorphisms $\mathbb{Z}/\langle 12 \rangle \to \mathbb{Z}/\langle 4 \rangle$ and $\mathbb{Z}/\langle 12 \rangle \to \mathbb{Z}/\langle 6 \rangle$.
3. If a product P exists in $\mathcal{D}$, then we have a morphisms $p_1 : P \to \mathbb{Z}/\langle 4 \rangle$ and $p_2 : P \to \mathbb{Z}/\langle 6 \rangle$ which are universal.
4. By universality, we have a morphism $\mathbb{Z}/\langle 12 \rangle \to P$ such that composing with p_i give the homomorphisms in (2).
5. By (1), this means that p_i are also surjective. This means that $P = \mathbb{Z}/\langle n \rangle$ with n divisible by 12 as discussed in the answer above.
6. We have the homomorphisms $\mathbb{Z}/\langle 2 \rangle \to \mathbb{Z}/\langle 4 \rangle$ (respectively, $\mathbb{Z}/\langle 2 \rangle \to \mathbb{Z}/\langle 6 \rangle$) given by sending 1 to 2 (respectively by sending 1 to 3). Clearly, there is no homomorphism $\mathbb{Z}/\langle 2 \rangle \to \mathbb{Z}/\langle 12 \rangle$ which gives these homomorphisms by composition.
7. Hence, $\mathbb{Z}/\langle 12 \rangle$ is *not* the product in $\mathcal{D}$. On the other hand, by (1)-(5), *if* the product exists it *has* to be $\mathbb{Z}/\langle 12 \rangle$.

Thus, we see that the product does not exist.