Adjoints and Products

As already seen, there is a close connection between adjoints and products. This becomes clearer with Freyd's adjoint functor theorem which we now explain and prove.

Adjoints and products for posets

Given a poset P, we can think of it as a category with a unique morphism between two distinct objects. A functor $f : P \to Q$ between posets considered as categories is the same as a monotone map of posets. Since morphisms betwen objects of P are unique when they exist, a diagram in P is *determined* by a suitable subset S of P which makes up the vertices of the diagram. It follows easily that the product of this diagram exists if and only if there exists a *greatest lower bound* $\inf(S)$ in P. This is an element of P defined by:

$$x \leq \inf(S) \text{ in } P \text{ if and only if } x \leq p, \text{ for all } p \in S$$

Similarly, the co-product of this diagram exists if and only if there exists a *least upper bound* $\sup(S)$. This is an element of P defined by:

$$\sup(S) \leq x \text{ in } P \text{ if and only if } p \leq x, \text{ for all } p \in S$$

The functor $f : P \to Q$ is a left adjoint to $g : Q \to P$ whenever the following holds.

$$f(x) \leq y \text{ in } Q \text{ if and only if } x \leq g(y) \text{ in } P$$

With the above observations, we can now try to understand the relation between products, co-products and adjoints for categories that are posets.

Right/Left adjoint preserves products/co-products

Now assume that $f : P \to Q$ is the left adjoint of $g : Q \to P$. This also means that g is the right adjoint of f. Given a subset S of Q such that $y_0 = \inf(S)$ exists in Q, we have

$$\begin{aligned} x \leq g(y_0) &\text{ if and only if } f(x) \leq y_0 \\ &\text{ if and only if } f(x) \leq y, \text{ for all } y \in S \\ &\text{ if and only if } x \leq g(y), \text{ for all } y \in S \end{aligned}$$

It follows that $g(y_0) = \inf g(S)$. In other words, we see that if y_0 is the product of S in Q, then $g(y_0)$ is the product of $g(S)$ in P.

Similarly, if $x_0 = \sup(T)$ is the co-product of a subset T in P, then we can see that $f(x_0)$ is the co-product of $f(T)$ in Q.

In other words, for functors between posets, left adjoint functors preserve co-products and right adjoint functors preserve products.

We will generalise this to other categories.

Constructing adjoints for posets

Conversely, suppose that P is a poset for which *every* subset has a supremum. As seen above, this is the same as saying that in this category diagrams have co-products. Moreover, assume that we have a functor $f : P \to Q$ which preserves these co-products. Equivalently, f is a (monotone) map of posets which satisfies

$$\sup f(T) = f\left(\sup(T)\right)$$

for every subset T in P. In other words, the least upper bound of $f(T)$ exists and is the image under f of the the least upper bound of T.

For each y in Q consider the subset $T_y = \{x \in P : f(x) \leq y\}$ in P. By assumption on P, the least upper bound of T_y exists. We denote $g(y) = \sup(T_y)$.

Since f preserves suprema $f(g(y)) = \sup(f(T_y))$. Now, x lies in T_y if and only if $f(x) \leq y$. Thus, $f(g(y)) = \sup(f(T_y)) \leq y$. In particular, $g(y)$ lies in T_y and is the largest element in it. Hence, if $f(x) \leq y$, then $x \in T_y$ so $x \leq g(y)$. Conversely, if $x \leq g(y)$, then $f(x) \leq f(g(y)) \leq y$. Thus, we see that g is the right adjoint of f.

In summary, if P is a poset that has least upper bounds and $f : P \to Q$ is a monotone map that preserves these least upper bounds, then there is a monotone map $g : Q \to P$ such that g is the right adjoint of f.

Similarly, if Q is a poset that has greatest lower bounds and $g : Q \to P$ is a monotone map that preserves these greatest lower bounds, then there is a monotone map $f : P \to Q$ such that f is the left adjoint of g.

We will generalise this to other categories.

Existence of all products only happens for posets

Suppose $\mathcal{C}$ is a small category such that for every small category $\mathcal{I}$ and functor $F : \mathcal{I} \to \mathcal{C}$, the product $\prod_{\mathcal{I}} F$ exists.

Recall that the product represents the (contravariant) functor $\overline{F}$ to **Set** given by

$$\overline{F}(X) = \mathrm{Nat}(\Delta X, F)$$

where $\Delta X : \mathcal{I} \to \mathcal{C}$ is the functor that maps all objects of $\mathcal{I}$ to X and all morphisms of $\mathcal{I}$ to the identity endomorphism of X, and $\mathrm{Nat}(K, L)$ denotes the natural transformations from a functor K to a functor L when both are functors between the same categories. This means that:

- We have a natural transformation $\pi : \Delta \prod_{\mathcal{I}} F \to F$
- The map $f \mapsto \pi \circ \Delta f$ gives a bijection

$$\mathcal{C}\left(X, \prod_{\mathcal{I}} F\right) \xrightarrow{\simeq} \mathrm{Nat}(\Delta X, F)$$

Now consider a set S as a category with only identity morphisms. This is a small category. Given an object Y in $\mathcal{C}$, we have a functor $\Delta Y : S \to \mathcal{C}$ as above. Since $\mathcal{C}$ has products over small categories, we have a product $\prod_S \Delta Y$ in $\mathcal{C}$ and it satisfies

$$\mathcal{C}\left(X, \prod_{\mathcal{S}} \Delta Y\right) \xrightarrow{\simeq} \mathrm{Nat}(\Delta X, \Delta Y)$$

Now $\mathcal{C}$ is a small category, so morphisms in it form a set C_1 and $\mathcal{C}\left(X, \prod_{\mathcal{S}} \Delta Y\right)$ is a subset of C_1.

For objects X and Y in $\mathcal{C}$, we note that natural transformations $\Delta X \to \Delta Y$ are given by a set map $S \to \mathcal{C}(X, Y)$ since there are *no* morphisms in S and thus no commutative diagrams to be satisfied.

It follows that $\mathrm{Map}(S, \mathcal{C}(X, Y))$ is in bijection with a subset of C_1 and this happens for *any* choice of set S! In particular, we can do this with $S = C_1$.

This is impossible, unless $\mathcal{C}(X, Y)$ is at most a singleton for each X and Y in $\mathcal{C}$. In other words, $\mathcal{C}$ is (essentially) a poset! (Essentially, in the sense that we identify isomorphic objects of the category since there is a unique isomorphism between any two isomorphic objects.)

> A small category $\mathcal{C}$ has *all* products over small categories only if it is essentially a poset.

By taking duals, we see that the same holds for co-products.

General categories

We now look at generalisations of the above results to general categories.

First of all, we should show that right adjoints preserve products and left adjoints preserve co-products.

Secondly, we would like to understand to what extent the property that a functor F preserves co-products implies that it has a right adjoint; i.e. that the functor F is the left-adjoint of some functor. Note that the category on which F acts needs to have *some* co-products, otherwise the condition that it preserves co-products is vacuous! On the other hand, as seen above, the condition that it contains *all* co-products is too restrictive.

Right adjoints preserve products

Given a category $\mathcal{C}$ and a functor $F : \mathcal{C} \to \mathcal{D}$ which is left-adjoint to a functor $G : \mathcal{D} \to \mathcal{C}$ such that $u : 1_{\mathcal{C}} \to GF$ is the unit and $v : FG \to 1_{\mathcal{D}}$ is the co-unit. The unit and co-unit yield a natural identification

$$\mathcal{C}(X, G(Y)) = \mathcal{D}(F(X), Y)$$

given by $a \mapsto v_Y \circ F(a)$ going left to right and $b \mapsto G(b) \circ u_X$ going right to left. In this section we use this identification without explicit mention of these maps.

We first show that G preserves products and F preserves co-products.

Given an indexing category $\mathcal{I}$ recall that an $\mathcal{I}$-schema, is a functor $E : \mathcal{I} \to \mathcal{D}$. If $\Delta(X) : \mathcal{I} \to \mathcal{D}$ denotes the "constant functor" X as above, then we defined the functor $\overline{E}$ from $\mathcal{D}^{\text{opp}}$ to **Set** as follows:

- Given an object X in $\mathcal{D}$ we associate the set $\overline{E}(X) = \text{Nat}(\Delta X, E)$ whose elements are natural transformations $\chi : \Delta X \to E$.
- Given a morphism $f : X \to Y$ in $\mathcal{D}$, we associate the set map $\overline{E}(Y) \to \overline{E}(X)$ given by $\eta \mapsto \eta \circ \Delta f$.

The product $\prod_{\mathcal{I}} E$ is a pair (P, π), where P is an object in $\mathcal{D}$ and π is a natural transformation $\pi : \Delta P \to E$ such that (P, π) represents the functor $\overline{E}$ so that

$$\mathcal{D}(X, P) \xrightarrow{\simeq} \overline{E}(X) = \text{Nat}(\Delta X, E) \text{ given by } f \mapsto \pi \circ \Delta f$$

is a bijection.

We now compose with the functor G, to get $(G(P), G(\pi))$ where $G(\pi)$ is a natural transformation $\Delta G(P) \to GE$. (Here both of these are considered as functors from $\mathcal{I}$ to $\mathcal{C}$.) We claim that this pair $(G(P), G(\pi))$ is the product of GE in $\mathcal{C}$. Note that:

- $\text{Nat}(\Delta W, GE)$ corresponds to morphisms $W \to GE(i)$ for each object i in $\mathcal{I}$.
- morphisms $W \to GE(i)$ correspond to morphisms $F(W) \to E(i)$ by adjunction.

It follows that we have an identification

$$\text{Nat}(\Delta W, GE) = \text{Nat}(\Delta F(W), E)$$

Combining this with the above identification applied to $X = F(W)$, we get an identification

$$\mathcal{D}(F(W), P) \xrightarrow{\simeq} \text{Nat}(\Delta W, GE)$$

Using adjunction again we have an identification of $\mathcal{C}(W, G(P))$ with $\mathcal{D}(F(W), P)$. Thus, we get an identification

$$\mathcal{C}(W, G(P)) \xrightarrow{\simeq} \text{Nat}(\Delta W, GE)$$

This shows that $G(P), G(\pi))$ represents the functor $\overline{GE}$. Thus it is the product $\prod_{\mathcal{I}} GE$ in $\mathcal{C}$.

In summary, we see that the right adjoint G preserves products.

Similarly, we have seen that co-product $\coprod_{\mathcal{I}} E$ in $\mathcal{C}$ co-represents the functor $\underline{E}$ from $\mathcal{C}$ to **Set** given as follows:

- Given an object X in $\mathcal{C}$ we associate the set $\underline{E}(X) = \text{Nat}(E, \Delta X)$ whose elements are natural transformations $\chi : E \to \Delta X$.

- Given a morphism $f : X \to Y$ in $\mathcal{C}$, we associate the set map $\underline{E}(X) \to \underline{E}(Y)$ given by $\eta \mapsto \Delta f \circ \eta$.

Given that (C, γ) co-represents this functor, C is an object in $\mathcal{C}$ and $\gamma : E \to \Delta C$ is a natural transformation such that

$$\mathcal{C}(C, Z) \xrightarrow{\simeq} \mathrm{Nat}(E, \Delta Z)$$

is a bijection given by $f \mapsto \Delta f \circ \gamma$.

As above, we show that $(F(C), F(\gamma))$ co-represents the functor $\underline{FE}$. This shows that F preserves co-products.

In other words, we see that the left adjoint F preserves co-products.

Existence of adjoints

Given a category $\mathcal{C}$ and a functor $F : \mathcal{C} \to \mathcal{D}$ we examine how can ensure the existence of a right adjoint.

Given an object B of $\mathcal{D}$, we defined a functor F_B as the composite $B \cdot F'$ going from $\mathcal{C}^{\mathrm{opp}}$ to **Set**, where F' is the functor F considered as a functor from $\mathcal{C}^{\mathrm{opp}}$ to $\mathcal{D}^{\mathrm{opp}}$. More explicitly,

- For an object X of $\mathcal{C}$ we have $F_B(X) = \mathcal{D}(F(X), B)$.
- For a morphism $f : X \to Y$ in $\mathcal{C}$ we have

$$F_B(Y) = \mathcal{D}(F(Y), B) \to \mathcal{D}(F(X), B) = F_B(X) \text{ given by } a \mapsto a \circ F(f)$$

We saw that, if F has a right adjoint $G : \mathcal{D} \to \mathcal{C}$ with co-unit $v : FG \to 1_{\mathcal{D}}$ then F_B is represented by $(G(B), v_B)$ where $v_B : FG(B) \to B$ is the instantiation of the natural transformation v at the object B in $\mathcal{D}$.

Conversely, supppose that for any object B in $\mathcal{D}$, the functor F_B is representable and let us *denote* the representing pair as $(G(B), v_B)$ where $v_B : FG(B) \to B$ is a morphism in $\mathcal{D}$. This means that for every object B in $\mathcal{D}$ and every object X in $\mathcal{C}$, we have a bijection

$$\mathcal{C}(X, G(B)) \xrightarrow{\simeq} \mathcal{D}(F(X), B) \text{ given by } f \mapsto v_B \circ F(f)$$

Given an object A in $\mathcal{C}$, we apply this to $B = F(A)$ and $X = A$. This gives a morphism u_A in $\mathcal{C}(A, GF(A))$ such that $v_{F(A)} \circ F(u_A) = 1_{F(A)}$.

Secondly, given a morphism $h : C \to B$ in $\mathcal{D}$ we apply this to $X = G(C)$ and the morphism $h \circ v_C : FG(C) \to B$ which is in $\mathcal{D}(FG(C), B)$. There is a unique morphism $h' : G(C) \to G(B)$ such that $h \circ v_C = v_B \circ F(h')$. We *define* $G(h) = h'$.

We easily check that this gives rise to a functor $G : \mathcal{D} \to \mathcal{C}$ and that this is a right-adjoint to F with unit given by u and co-unit given by v.

We conclude that:

If, for every object B of $\mathcal{D}$, the functor F_B is representable, then F has a right adjoint G and $(G(B), v_B : FG(B) \to B)$ is the representing object in $\mathcal{C}$.

We can similarly work with the functor $F^B(X) = \mathcal{D}(B, F(X))$ to show that:

If, for every object B of $\mathcal{D}$, the functor F^B is representable, then F has a left adjoint K and $(K(B), u_B : B \to FK(B))$ is the representing object in $\mathcal{C}$.

We now look for conditions on C that can help us ensure that the functor F_B (or F^B as appropriate) is representable.

We first need to understand co-products in a special case.

Product/Co-product where the indexing category has an initial/final object

Given any functor $E : \mathcal{I} \to \mathcal{G}$. Recall, that the co-product $\coprod_{\mathcal{I}} E$ is a pair (C, γ) where $\gamma \in \mathrm{Nat}(E, \Delta C)$ which represents the functor $\underline{E}$:

- Given an object X in $\mathcal{G}$ we associate the set $\underline{E}(X) = \mathrm{Nat}(E, \Delta X)$ whose elements are natural transformations $\chi : E \to \Delta X$.
- Given a morphism $f : X \to Y$ in $\mathcal{C}$, we associate the set map $\underline{E}(X) \to \underline{E}(Y)$ given by $\eta \mapsto \Delta f \circ \eta$.

This means that we have a bijection

$$\mathcal{C}(C, Z) \xrightarrow{\simeq} \mathrm{Nat}(E, \Delta Z) \text{ given by } f \mapsto \Delta f \circ \gamma$$

If $\mathcal{I}$ has a final object I_0, for any object I in $\mathcal{I}$, we have a *unique* morphism $\iota_I : I \to I_0$. This gives a morphism $E(\iota_I) : E(I) \to E(I_0)$. Putting this together for objects I of $\mathcal{I}$ this defines a natural transformation $\iota : E \to \Delta E(I_0)$.

Given $\xi : E \to \Delta Z$, we put $f = \xi_{I_0} : E(I_0) \to Z$ and check that $\xi = \Delta f \circ \iota$. This shows that $(E(I_0), \iota)$ is the co-product $\coprod_{\mathcal{I}} E$. In summary:

If $\mathcal{I}$ has a final object I_0 and $E : \mathcal{I} \to \mathcal{G}$ is a functor, then $E(I_0)$ is the co-product $\coprod_{\mathcal{I}} E$.

Dually, we can show the following.

If $\mathcal{I}$ has an initial object I_0 and $E : \mathcal{I} \to \mathcal{G}$ is a functor, then $E(I_0)$ is the product $\prod_{\mathcal{I}} E$.

Co-product and Representability

Given a functor F from $\mathcal{C}$ to $\mathcal{D}$ and an object B of $\mathcal{D}$. In an earlier section we defined the category $F \downarrow B$ as follows.

- An object is a pair (X, h) where X is an object in $\mathcal{C}$ and $h : F(X) \to B$ is a morphism in $\mathcal{D}$.

- A morphism $f : (X, h) \to (Y, g)$ is a morphism $f : X \to Y$ in $\mathcal{C}$ such that $h = g \circ F(f)$.

We have a natural forgetful functor H_B from $F \downarrow B$ to $\mathcal{C}$. We now examine the relation between the representability of F_B and the existence of a co-product $\coprod_{F\downarrow B} H_B$.

If the functor F_B introduced above is represented by $(A, \alpha : F(A) \to B)$, then (A, α) is an object in $F \downarrow B$. Moreover, for any object X in $\mathcal{C}$ the map

$$\mathcal{C}(X, A) \overset{\simeq}{\to} \mathcal{D}(F(X), B) = F_B(X) \text{ given by } f \mapsto \alpha \circ F(f)$$

is a bijection. It follows that for every object (X, h) in $F \downarrow B$ there is a *unique* $f : X \to A$ such that $h = \alpha \circ F(f)$; this means that $f : (X, h) \to (A, \alpha)$ is a morphism in $F \downarrow B$. In other words, (A, α) is a final object in the category $F \downarrow B$. This morphism $(X, h) \to (A, \alpha)$ for every object (X, h) in $F \downarrow B$ defines the natural transformation $\iota : H_B \to \Delta A$. As seen above (A, ι) is the co-product $\coprod_{F\downarrow B} H_B$.

In general the co-product is given by an object (C, γ), where C is an object of $\mathcal{C}$ and $\gamma : H_B \to \Delta C$ is a natural transformation of functors from $F \downarrow B$ to $\mathcal{C}$ so that we have a bijection

$$\mathcal{C}(A, Z) \overset{\simeq}{\to} \mathrm{Nat}(H_B, \Delta Z) \text{ given by } f \mapsto \Delta f \circ \gamma$$

We now compose F with H_B to obtain a functor from $F \downarrow B$ to $\mathcal{D}$. This associates to every object (X, h) of $F \downarrow B$, the object $F(X)$ in $\mathcal{D}$. The *given* morphism $h : F(X) \to B$, thus provides a natural transformation $\phi : FH_B \to \Delta B$.

If we *assume* that F preserves co-products, then $(F(C), F(\gamma))$ is the co-product $\coprod_{F\downarrow B} FH_B$. Thus, we have morphism $c : F(C) \to B$ such that $\phi = \Delta c \circ F(\gamma)$. In particular, (C, c) is an object of $F \downarrow B$. One then checks that (C, c) is a *final* object of $F \downarrow B$ and that $\gamma : H_B \to \Delta C$ is the natural transformation associated with this final object. Hence F_B is represented by (C, c).

> In summary, if the co-product of the forgetful functor $F \downarrow B \to \mathcal{C}$ exists *and* this co-product is preserved by F, then the co-product also represents the functor F_B.

This shows that:

> If the co-product $G(B) = \coprod_{F\downarrow B} H_B$ exists for *every* object B in $\mathcal{D}$ and F preserves co-products, then G gives a right adjoint functor $G : \mathcal{D} \to \mathcal{C}$ of the functor F.

Similarly, we had introduced the category $B \downarrow F$ whose objects are pairs $(X, h : B \to F(X))$ and we have a forgetful functor H^B from $B \downarrow F$ to $\mathcal{C}$. We use arguments similar to the ones above to show:

> If the product $K(B) = \prod_{B\downarrow F} H^B$ exists for *every* object B in $\mathcal{D}$ and F preserves products, then K gives a left adjoint functor $K : \mathcal{D} \to \mathcal{C}$ of the functor F.

The “General Adjoint Functor Theorem” and the “Special Adjoint Functor Theorem” give some easier to check conditions on F and $\mathcal{D}$ which ensure the existence of the co-products (or products as appropriate) required in the above statements.