Revisiting Yoneda Lemma

Given a category $\mathcal{C}$, we defined the functor $A^{\cdot}$ from $\mathcal{C}^{\mathrm{opp}}$ to **Set** as follows:

- $A^{\cdot}(X) = \mathcal{C}(X, A)$ for an object X in $\mathcal{C}$.
- $A^{\cdot}(f)(a) = a \circ f$ giving $A^{\cdot}(f) : A^{\cdot}(Y) \to A^{\cdot}(X)$ for a morphism $f : X \to Y$ in $\mathcal{C}$.

Similarly, we defined the functor $A.$ from $\mathcal{C}$ to **Set** as follows:

- $A.(X) = \mathcal{C}(A, X)$ for an object X in $\mathcal{C}$.
- $A.(f)(a) = f \circ a$ giving $A.(f) : A.(X) \to A.(Y)$ for a morphism $f : X \to Y$ in $\mathcal{C}$.

Representability and Co-representability

If F is a functor from $\mathcal{C}^{\mathrm{opp}}$ to **Set**, note that for an element $a \in A^{\cdot}(X) = \mathcal{C}(X, A)$, we get a set map $F(a) : F(A) \to F(X)$. Thus, we get a pairing

$$F(A) \times A^{\cdot}(X) \to F(X) \text{ given by } (\alpha, a) \mapsto F(a)(\alpha)$$

Fixing $\alpha \in F(A)$, this allows us to define, for each object X, a map $\tilde{\alpha}_X : A^{\cdot}(X) \to F(X)$ given by $a \mapsto F(a)(\alpha)$. We see easily that this gives a natural transformation $\tilde{\alpha} : A^{\cdot} \to F$. Conversely, given a natural transformation $\eta : A^{\cdot} \to F$, we can define α as the image of 1_A under $\eta_A : A^{\cdot}(A) \to F(A)$ and check that $\eta = \tilde{\alpha}$.

In summary, we have an natural identification between elements $\alpha \in F(A)$ and natural transformations $\tilde{\alpha} : A^{\cdot} \to F$. This is the Yoneda lemma for contravariant functors F.

We say that F is *represented by* (A, α) if this natural transformation is an isomorphism of functors; equivalently, this means that $\tilde{\alpha}_X : \mathcal{C}(X, A) \to F(X)$ is a bijection for each object X in $\mathcal{C}$.

Similarly, given a functor F from $\mathcal{C}$ to **Set** we have a pairing

$$F(A) \times A.(X) \to F(X) \text{ given by } (\alpha, a) \mapsto F(a)(\alpha)$$

since $a \in A.(X) = \mathcal{C}(A, X)$ gives a set map $F(a) : F(A) \to F(X)$. By repeating the argument above with minor modifications we see that this gives a natural identification between elements $\alpha \in F(A)$ and natural transformations $\tilde{\alpha} : A. \to F$. This is the Yoneda lemma for (covariant) functors F.

We say that F is *co-represented by* (A, α) if this natural transformation is an isomorphism of functors; equivalently, this means that $\tilde{\alpha}_X : \mathcal{C}(A, X) \to F(X)$ is a bijection for each object X in $\mathcal{C}$.

Universals represent functors

We will now see that universal objects can be seen as representing and co-representing functors.

To begin this discussion consider the functor U that sends every object of $\mathcal{C}$ to the singleton set $\{\cdot\}$ and every morphism in $\mathcal{C}$ to the identity map $1_{\{\cdot\}}$ of this singleton set. Note that this is also a functor from $\mathcal{C}^{\mathrm{opp}}$ to **Set**.

What can we say about representability and co-representability of U?

If (A, α) represents U, then $\alpha = \cdot$ is the unique element of $U(A) = \{\cdot\}$ *and* for every object X in $\mathcal{C}$, this gives a bijection $\tilde{\alpha} : \mathcal{C}(X, A) \to U(X) = \{\cdot\}$. This means that there is a *unique* morphism $X \to A$ for every object X in $\mathcal{C}$. In other words,

> If A represents the singleton functor U, then A is a final object in $\mathcal{C}$.

Similarly, if (B, β) co-represents U, then β unique element of $U(B) = \{\cdot\}$ *and* for every objection X in $\mathcal{C}$, this gives a bijection $\tilde{\beta} : \mathcal{C}(B, X) \to U(X) = \{\cdot\}$. This means that there is a *unique* morphism $B \to X$ for every object X in $\mathcal{C}$. In other words,

> If B co-represents the singleton functor U, then B is an initial object in $\mathcal{C}$.

Products and Limits of Schemas

Given a category $\mathcal{C}$, we saw that a diagram D in $\mathcal{C}$ based on a directed graph Γ is described precisely by a functor $F_D : \mathbf{P}_\Gamma \to \mathcal{C}$, where $\mathbf{P}_\Gamma$ is the category where objects are vertices in Γ and morphisms are directed paths in Γ.

More generally, given a (small) category $\mathcal{I}$, we define an $\mathcal{I}$-schema in $\mathcal{C}$ to be a functor $F : \mathcal{I} \to \mathcal{C}$.

Given an object X in $\mathcal{C}$, we denote by ΔX the functor from $\mathcal{I}$ to $\mathcal{C}$ which sends every object of $\mathcal{I}$ to X and every morphism in $\mathcal{I}$ to the identity morphism 1_X. Note that this makes sense independent of the category $\mathcal{I}$, so we use the same notation ΔX without worrying about the category $\mathcal{I}$.

We noted that a morphism from X to the diagram D is described precisely by a natural transformation $\chi : \Delta X \to F_D$. Similarly, a morphism from the diagram D to Z is precisely a natural transformation $\xi : F_D \to \Delta Z$. We then looked at the category of pairs (X, χ). A final object in this category, if it exists, is precisely the product $\prod D$. Similarly, if there is an initial object in the category of pairs (Z, ξ), it is the co-product $\coprod D$.

More generally, we can consider the category of pairs (X, χ) where $\chi : \Delta X \to F$ is a natural transformation of functors $\mathcal{I}$ to $\mathcal{C}$ where morphisms $(X, \chi) \to (Y, \eta)$ are morphisms $f : X \to Y$ in $\mathcal{C}$ that yield commutative diagrams

$$\begin{array}{ccc} \Delta X & \xrightarrow{\Delta f} & \Delta Y \\ {\scriptstyle\chi}\downarrow & & \downarrow{\scriptstyle\eta} \\ F & = & F \end{array}$$

where $\Delta f : \Delta X \to \Delta Y$ is the natural transformation that associates f to every object of $\mathcal{I}$. In other words, we require $\eta \circ \Delta f = \chi$ for $f : X \to Y$ to yield a morphism $(X, \chi) \to (Y, \eta)$.

We then define the product $(\prod_{\mathcal{I}} F, \pi)$ of the $\mathcal{I}$-schema F in $\mathcal{C}$ as the final object in this category, if it exists.

Similar, we consider the category of pairs (Z, ξ) where $\xi : F \to \Delta Z$ is a natural transformation of functors $\mathcal{I}$ to $\mathcal{C}$, where morphisms $(Z, \xi) \to (W, \omega)$ are given by morphisms $f : Z \to W$ such that $(\Delta f) \circ \xi = \omega$.

We then define the co-product $(\coprod_{\mathcal{I}} F, \iota)$ of the $\mathcal{I}$-schema F in $\mathcal{C}$ as the initial object in this category, if it exists.

We now exhibit these in terms of representation and co-representation of functors.

Functors associated with Schemas

Given a $\mathcal{I}$-scheme F in $\mathcal{C}$. (Note that this is another name for a functor F from $\mathcal{I}$ to $\mathcal{C}$!)

We define a functor $\overline{F}$ from $\mathcal{C}^{\mathrm{opp}}$ to **Set** as follows:

- Given an object X in $\mathcal{C}$ we associate the set $\overline{F}(X)$ whose elements are natural transformations $\chi : \Delta X \to F$.
- Given a morphism $f : X \to Y$ in $\mathcal{C}$, we associate the set map $\overline{F}(Y) \to \overline{F}(X)$ given by $\eta \mapsto \eta \circ \Delta f$.

If (A, α) represents the functor $\overline{F}$, then $\alpha : \Delta A \to F$ is a natural transformation such that the for every object X in $\mathcal{C}$,

$$f \to \alpha \circ \Delta f \text{ gives a bijection } A^{\cdot}(X) = \mathcal{C}(X, A) \to \overline{F}(X)$$

Put differently, given a $\chi : \Delta X \to F$, there is a unique $f : X \to A$ such that $\chi = \alpha \circ \Delta f$. This is precisely the same as saying that (A, α) is the product $(\prod_{\mathcal{I}} F, \pi)$.

Similarly, we define a functor $\underline{F}$ from $\mathcal{C}$ to **Set** as follows:

- Given an object Z in $\mathcal{C}$ we associate the set $\underline{F}(Z)$ whose elements are natural transformations $\xi : F \to \Delta Z$.
- Given a morphism $f : Z \to W$ in $\mathcal{C}$, we associate the set map $\underline{F}(Z) \to \underline{F}(W)$ given by $\xi \mapsto (\Delta f) \circ \xi$.

If (B, β) co-represents the functor $\underline{F}$, then $\beta : F \to \Delta B$ is a natural transformation such that the for every object Z in $\mathcal{C}$,

$$f \to (\Delta f) \circ \beta \text{ gives a bijection } B.(X) = \mathcal{C}(B, Z) \to \underline{F}(Z)$$

Put differently, given a $\xi : F \to \Delta Z$, there is a unique $f : B \to Z$ such that $\xi = (\Delta f) \circ \beta$. This is precisely the same as saying that (B, β) is the co-product $(\coprod_{\mathcal{I}} F, \iota)$.

Adjoint functors and representability

Given a functor $F : \mathcal{C} \to \mathcal{D}$ and an object A in $\mathcal{D}$, we ask for the representability of the functor F_A from $\mathcal{C}^{\text{opp}}$ to **Set** defined as follows:

- For an object X of $\mathcal{C}$ we define $F_A(X) = \mathcal{D}(F(X), A)$.
- For a morphism $f : X \to Y$ in $\mathcal{C}D$ we define

$$F_A(Y) = \mathcal{D}(F(Y), A) \to \mathcal{D}(F(X), A) = F_A(X) \text{ given by } a \mapsto a \circ F(f)$$

Note that F can also be seen as a functor $\mathcal{C}^{\text{opp}}$ to $\mathcal{D}^{\text{opp}}$ in an obvious way; let us denote this functor as F'. We then check that F_A is the composite functor $A^{\cdot}F'$.

For (Z, z) to represent this functor, the following conditions must hold.

- Z is an object in $\mathcal{C}$ and $z : F(Z) \to A$ is a morphism in $\mathcal{D}$.
- For an object X in $\mathcal{C}$, we have a bijection

$$Z^{\cdot}(X) = \mathcal{C}(X, Z) \to \mathcal{D}(F(X), A) = F_A(X) \text{ given by } f \mapsto z \circ F(f)$$

If $G : \mathcal{D} \to \mathcal{C}$ is a *right adjoint* to F, and $v_A : FGA \to A$ is the co-unit at the object A of $\mathcal{D}$, we see that (GA, v_A) represents the functor F_A.

Similarly, we can define the functor F^A from $\mathcal{C}$ to **Set** as follows:

- For an object X of $\mathcal{C}$ we define $F^A(X) = \mathcal{D}(A, F(X))$.
- For a morphism $f : X \to Y$ in $\mathcal{C}D$ we define

$$F^A(X) = \mathcal{D}(A, F(X)) \to \mathcal{D}(A, F(Y)) = F^A(Y) \text{ given by } a \mapsto F(f) \circ a$$

We can check that $F^A = A.F$ is the composite functor.

For (W, w) to co-represent this functor, the following conditions must hold.

- W is an object in $\mathcal{C}$ and $w : A \to F(W)$ is a morphism in $\mathcal{D}$.
- For an object X in $\mathcal{C}$, we have a bijection

$$W.(X) = \mathcal{C}(W, X) \to \mathcal{D}(A, F(X)) = F^A(X) \text{ given by } f \mapsto F(f) \circ w$$

If $H : \mathcal{D} \to \mathcal{C}$ is a *left adjoint* to F, and $u_A : A \to FHA$ is the unit at the object A of $\mathcal{D}$, we see that (HA, u_A) co-represents the functor F^A.

We thus see that representable/co-representable functors are a way to interpret right/left adjoints "object-wise".