# Quiz 5: Epic and monic morphisms

### Question

Consider the category $\mathcal{C}$ with objects as non-negative integers and morphisms as matrices with integer entries so that composition is given by matrix multiplication. Give an example (or prove that one does not exist) of a monic morphism that has no retraction, an epic morphism that has no section and a morphism that is epic and monic but is not an isomorphism.

### Answer

One hint is that the category is essentially equivalent to the category of finitely generated free abelian groups. Another observation is that the rank of an integer matrix is the same as the rank of the same matrix considered as a matrix with rational entries.

Let $M$ be a $p \times q$ integer matrix that represents a monomorphism in $\mathcal{C}$. This means that for all $r$ and all $q \times r$ matrices $N_1$ and $N_2$, if we have $MN_1 = MN_2$ then $N_1 = N_2$. Equivalently, this means that for any $q \times r$ matrix $N$, if $MN = 0$ then $N = 0$. In particular, for a column vector $v$ of length $q$, we have $Mv = 0$ if and only if $v = 0$. This is the same as saying that the columns of $M$ are linearly independent, which means that $M$ has rank $q$.

We conclude that if $M$ is a $p \times q$ integer matrix, then the morphism given by it in $\mathcal{C}$ is an monomorphism if and only if $M$ has rank $q$.

Let $M$ be a $p \times q$ integer matrix that represents an epimorphism in $\mathcal{C}$. This means that for all $r$ and all $r \times p$ matrices $N_1$ and $N_2$ if we have $N_1M = N_2M$, then $N_1 = N_2$. Equivalently, this means that for any $r \times q$ matrix $N$, if $NM = 0$ then $N = 0$. As above (or by taking transpose of the equation!), we see that this means that the rows of $M$ are linearly independent, which means that $M$ has rank $p$.

We conclude that if $M$ is a $p \times q$ integer matrix, then the morphism given by it in $\mathcal{C}$ is an epimorphism if and only if $M$ has rank $p$.

Let $M$ be a $p \times q$ integer matrix that represents an isomorphism in $\mathcal{C}$. Then $p = q$ and there is an $p \times p$ *integer* matrix $N$ such that $MN$ and $NM$ are both identity matrices.

From the above observations, we see that the $1 \times 1$ matrix with entry 2 is monic and epic and *not* an isomorphism. It follows easily that this has no retraction or section.