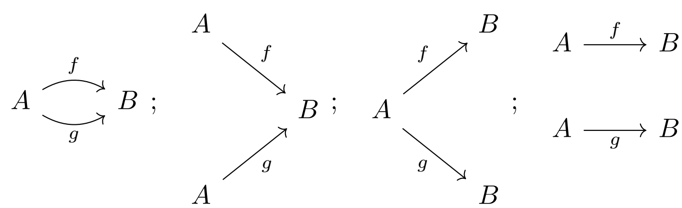
# Diagrams, products and co-products

A diagram $D$ in a category $\mathcal{C}$ is a directed graph where the vertices are objects of $\mathcal{C}$ and the arrows are morphisms of $\mathcal{C}$. For example:

- The empty diagram $\emptyset$ has no morphisms.
- Given a collection $S$ of objects we can take the diagram $S$ which consists of these objects as vertices and no arrows/morphisms.
- Given two morphisms $f, g : A \to B$, we have at least four diagrams possible

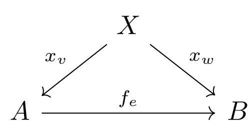



Note that we allow a single object to appear as multiple vertices in $D$ and a single morphism to appear as multiple arrows in $D$.

### Products

One question we can ask is whether there is an object $X$ in $\mathcal{C}$ that maps to the diagram $D$ in the following sense.

- For each vertex $v$ of $D$, if $A_v$ is the associated object, we have a morphism $x_v : X \to A_v$.
- For each (directed) edge $e$ from the vertex $v$ to the vertex $w$ of $D$, if $f_e : A_v \to A_w$ is the associated morphism, we have $x_w = f_e \circ x_v$.




We can think of the collection $(x_v)_{v \in D}$ as:

> A cone of arrows emerging from $X$ and ending at vertices in $D$ such that the triangles formed by these arrows with arrows of $D$ commute.

We denote such a collection by $(X, (x_v)_{v \in D})$.

We can further ask is whether there is a "largest" such collection $(\prod D, (p_v)_{v \in D})$ in the following sense:

- $(\prod D, (p_v)_{v \in D})$ maps to the diagram $D$ in the above sense.
- If $(X, (x_v)_{v \in D})$ maps to the diagram $D$ in the above sense, then there is a *unique* morphism $x : X \to \prod D$ such that $x_v = p_v \circ x$.

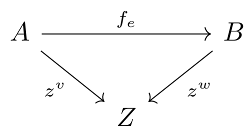
When such a $(\prod D, (p_v)_{v \in D})$ exists, we say that $(\prod D, (p_v)_{v \in D})$ is the product of the diagram $D$.

### Co-products

We can reverse all arrows in this description to obtain the notion of co-product of the diagram $D$.

We first ask if there is an object $Z$ in $\mathcal{C}$ to which the diagram $D$ maps in the following sense:

- For each vertex $v$ of $D$, if $A_v$ is the associated object, we have a morphism $z^v : A_v \to Z$.
- For each (directed) edge $e$ from the vertex $v$ to the vertex $w$ of $D$, if $f_e : A_v \to A_w$ is the associated morphism, we have $z^v = z^w \circ f_e$.




We can think of the collection $(z^v)_{v \in D}$ as:

> A cone of arrows emerging from vertices in $D$ and ending at $Z$ such that the triangles formed by these arrows with arrows of $D$ commute.

We can then ask whether there is a "smallest" such collection $(\coprod D, (c^v)_{v \in D})$ in the following sense:

- The diagram $D$ maps to $(\coprod D, (c^v)_{v \in D})$ in the above sense.
- If the diagram $D$ maps to $Z$ in the above sense then there is a *unique* morphism $z : \coprod D \to Z$ such that $z^v = z \circ c^A$.

When such a $(\coprod D, (c^v)_{v \in D})$ exists, we say that $(\coprod D, (c^v)_{v \in D})$ is the co-product of the diagram $D$.

## Examples

We now look at some examples of diagrams and associated products and co-products.

### Posets

Let $P$ be a poset considered as a category. Two objects of $P$ are comparable if and only if there is a morphism from one to the other and such a morphism is unique and has a unique direction.

It follows easily that, for the purposes of calculating products and co-products, giving a diagram $D$ in $P$ is the same as giving its collection $S$ of vertices; the morphisms are irrelevant.

Further, we see that the product is the *infimum* or "greatest lower bound" of $S$ and the co-product is the *supremum* or least upper bound of $S$.

### Initial and Final objects

When $D = \emptyset$ there are no conditions.

So what we are asking is whether is an object $\prod \emptyset$ such that given an object $A$ in $\mathcal{C}$, there is a *unique* morphism $A \to \prod \emptyset$.

If such an object exists we usually denote it as 1 or $1_{\mathcal{C}}$ instead of $\prod \emptyset$ and call it a *final object* of the category $\mathcal{C}$.

> We say 1 is a final object of $\mathcal{C}$ if, for every object $A$ of $\mathcal{C}$, there is a *unique* morphism $A \to 1$.

Similarly, if the co-product $\coprod \emptyset$ exists, we denote it by 0 or $0_{\mathcal{C}}$ and call it an *initial object* of the category $\mathcal{C}$.

> We say 0 is a final object of $\mathcal{C}$ if, for every object $A$ of $\mathcal{C}$, there is a *unique* morphism $0 \to A$.

Note that the unique morphism $1 \to 1$ *has to be* the identity morphism $1_1$. It follows easily that if $1'$ is *also* a final object, then the unique morphisms $1' \to 1$ and $1 \to 1'$ as above are inverses of each other.

> The final object, if it exists, is unique upto a unique isomorphism.

Similarly:

> The initial object, if it exists, is unique upto a unique isomorphism.

We also similarly conclude that any morphism $1 \to A$ has $A \to 1$ as a retraction, and any morphism $A \to 0$ has $0 \to A$ as a section.

### Diagram with two isolated vertices

Next, we examine the case where $D = \{v, w\}$ consists of two vertices associated with objects $A_v$ and $A_w$ (which may be the same!).

In this case, for the product, we are asking for objects $X$ with morphisms $x_v : X \to A_v$ and $x_w : X \to A_w$. (Even if $A_v$ and $A_w$ are the *same* object in $\mathcal{C}$, we can have *different* morphisms $x_v$ and $x_w$!) There is no further condition on $x_v$ or $x_w$.

The product if it exists in this case is denoted as $A_v \times A_w$. If so, have two morphisms $p_v : A_v \times A_w \to A_v$ and $p_w : A_v \times A_w \to A_w$. Moreover, given $(X, x_v, x_w)$ as above, there is a morphism $x : X \to A_v \times A_w$ such that $x_v = p_v \circ x$ and $x_w = p_w \circ x$.

Similarly, for the co-product, we are asking for objects $Z$ with morphisms $z^v : A_v \to Z$ and $z^w : A_w \to Z$. There is no further condition on $z^v$ or $z^w$.

The co-product if it exists in this case is denoted as $A_v \coprod A_w$. If so, have two morphisms $i^v : A_v \to A_v \coprod A_w$ and $i^w : A_w \to A_v \coprod A_w$. Moreover, given

$(Z, z^v, z^w)$ as above, there is a morphism $z : A_v \coprod A_w \to Z$ such that $z^v = z \circ i^v$ and $z^w = z \circ i^w$.

The notation $A_v \times A_w$ and $A_v \coprod A_w$ comes from the category of sets, where the product is the product of sets and the co-product is the disjoint union.

Note that, in terms of the functors $A^{\cdot}$ from $\mathcal{C}^{\text{opp}}$ to **Set** we have

$$A^{\cdot}_v(X) \times A^{\cdot}_w(X) = (A_v \times A_w)^{\cdot}(X)$$

for every object $X$ in $\mathcal{C}$. This can be seen as another definition of the product.

### Equalizer and co-equalizer

Given a pair $f, g : A \to B$ of morphisms in $\mathcal{C}$ we consider the diagram $D$ with two vertices and two arrows:

$$v \underset{e}{\overset{d}{\rightrightarrows}} w \quad \text{which is represented by} \quad A \underset{g}{\overset{f}{\rightrightarrows}} B$$

The product $\prod D$ is called the *equalizer* of $f$ and $g$.

More directly, we are looking for morphisms $x_A : X \to A$ such that $f \circ x_A = g \circ x_A$.

The equaliser $E(f = g)$ has a morphism $e : E(f = g) \to A$ such that $f \circ e = g \circ e$. Moreover, given $(X, x_A)$ as above, there is a unique morphism $x : X \to E(f = g)$ such that $x_A = e \circ x$.

If $\mathcal{C}$ is the category of sets, then $f$ and $g$ are set maps and the equaliser is the subset $\{a \in A | f(a) = g(a)\}$ of $A$.

In the category **Grp** of groups, the equaliser of two homomorphisms $f, g : G \to K$ is the subgroup $H$ of $G$ where these two homomorphisms become equal. In particular, if $g = e$ is the homomorphism which maps every element of $G$ to the identity element of $K$, then the equaliser is the kernel of $f$. Thus, we see that the notion of equaliser generalises the notion of kernels of homomorphisms of groups.

Daully, the co-product $\coprod D$ of the diagram above is called the *co-equalizer* of $f$ and $g$.

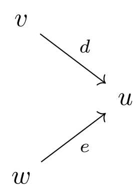
We are looking for morphisms $z^B : B \to Z$ such that $z^B \circ f = z^C \circ g$.

The co-equaliser $C(f = g)$ has a morphism $c : B \to C(f = g)$ such that $c \circ f = c \circ g$. Moreover, given $(Z, z^B)$ as above, there is a unique morphism $z : C(f = g) \to Z$ such that $z^B = z \circ c$.

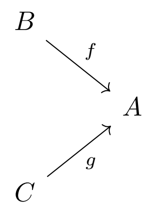
In the category **Ab** of Abelian groups, co-equaliser of homomorphisms $f, g : A \to B$ is precisely the cokernel of $f - g$. Thus, the notion of co-equaliser generalises the notion of cokernels of homomorphisms of abelian groups.

**Fibre products**

Given morphisms $f : B \to A$ and $g : C \to A$, we consider the diagram $D$ with three vertices and two arrows:




which is represented by




The product $\prod D$ is called the *fibre product* of $f$ and $g$ and is denoted as $B \times_A C$ (or sometimes $B \,{}_f\!\times_g C$ in order to make the morphisms $f$ and $g$ explicit).

We are asking for and object $X$ and morphisms $x_B : X \to B$ and $x_C : X \to C$ such that $f \circ x_B = g \circ x_C$.

The fibre product $B \times_A C$ has such morphisms $q_B : B \times_A C \to B$ and $q_C : B \times_A C \to C$. Moreover, given $(X, x_B, x_C)$ as above, there is a unique morphism $x : X \to B \times_A C$ such that $x_B = q_B \circ x$ and $x_C = q_C \circ x$.

In the category **Set**, we see that

$$B \times_A C = \{(b, c) | f(b) = g(c)\}$$

is a subset of $B \times C$. In fact, it is the equaliser of $f \circ p_B$ and $g \circ p_C$ where $p_B : B \times C \to B$ and $p_C : B \times C \to C$ are the natural projections.

More generally, if the product $B \times C$ is defined in a category $\mathcal{C}$, then the equaliser of $f \circ p_B$ and $g \circ p_C$ is precisely the fibre product of $f$ and $g$ as defined above.

If the category $\mathcal{C}$ has a final object 1, we note that there is a unique morphism from any object to 1. It follows easily that that $B \times C = B \times_1 C$.

**Joins or Amalgams**

By dualising the notion of fibre products, we get the notion of *joins* or *amalgams* of $B$ and $C$ along $f : A \to B$ and $g : A \to C$.

We are looking for objects $Z$ with morphisms $z^B : B \to Z$ and $z^C : C \to X$ such that $z^B \circ f = z^C \circ g$ as morphisms $A \to X$.

The join $B *_A C$ is such an object with morphisms $j^B : B \to B *_A C$ and $j^C : C \to B *_A C$. Moreover, givem $(Z, z^B, z^C)$ as above, there is a unique morphism $z : B *_A C \to Z$ such that $z^B = z \circ j^B$ and $z^C = z \circ j^C$.

In the category **Set** of sets, $B *_A C$ can be seen as the quotient of the disjoint union $B \coprod C$ obtained by identifying $b$ and $c$ if there is an $a$ such that $f(a) = b$ and $g(a) = c$.

If the category $\mathcal{C}$ has an initial object 0, then one can see that $A \coprod B$ is the amalgam $A *_0 B$ of the natural morphisms $0 \to A$ and $0 \to B$.

By dualising the statements of the previous section regarding products, equalisers and fibre products, we se that if the category $\mathcal{C}$ has co-products and equalisers then it has amalgams.