Quiz 3: Adjunction

Question: Given the functor G from $\mathcal{D}$ to $\mathcal{C}$ as follows, describe the left adjoint functor F.

The category $\mathcal{D}$ has objects as positive integers and morphisms are from n to nk for positive integers n and k.

The category $\mathcal{C}$ has objects as positive rational numbers and morphisms are from r to rk for a positive rational r and a positive integer k.

The functor G associates to a positive integer n, the same number considered as a rational number n and to a morphism $n \to nk$ in $\mathcal{D}$, the morphism $n \to nk$ in $\mathcal{C}$.

Answer: There is a morphism $r \to n = Gn$ in $\mathcal{D}$ if and only if $n = rk$ for some positive integer k.

We need the equality $\mathcal{C}(r, Gn) = \mathcal{D}(Fr, n)$. Both of these are either singleton sets or empty sets.

Thus, given a rational number r we want Fr to be a positive integer such that if n is another other positive integer, then $n = (Fr)k'$ if and only if $n = rk$ for some positive integers k' and k.

Writing $r = p/q$ where $\gcd(p, q) = 1$ and p and q are positive integers, we see that $rk = pk/q$ is an integer if and only of $k = qk'$ is a multiple of q. It follows that $Fr = p = rq$ solves the problem.

We note that the natural transformation $r \to GFr$ is the one given by $r \to rq = p$. The natural transformation $n = FGn \to n = n \cdot 1$ is the identity map.