Representable functors and Yoneda Lemma

Given an object A in a category $\mathcal{C}$, we can think of morphisms $b : X \to A$ as "generalised elements of A of type X", or more simply "X-points of A". This way of thinking allows us to think about objects as "made up of elements" in a manner similar to how we work with set-theoretical mathematical objects.

When morphisms between pairs A, B, of objects in $\mathcal{C}$ form a set $\mathcal{C}(A, B)$, this leads us to associate to A, the functor $A^{\cdot}$ from $\mathcal{C}^{\text{opp}}$ to **Set** given as follows:

- Given an object X, we associate the set $A^{\cdot}(X) = \mathcal{C}(X, A)$.
- Given a morphism $f : X \to Y$, we associate the set map

$$A^{\cdot}(f) : A^{\cdot}(Y) = \mathcal{C}(Y, A) \to \mathcal{C}(X, A) = A^{\cdot}(X)$$

  given by $(y : Y \to A) \mapsto (y \circ f : X \to A)$

The fact that this is a functor follows from the identity and associativity of morphisms in $\mathcal{C}$.

In the rest of this section we will work with a category $\mathcal{C}$ in which morphisms between pairs A, B of objects form a set $\mathcal{C}(A, B)$.

Representable functors

Given a functor F from $\mathcal{C}^{\text{opp}}$ to **Set** we can ask whether it is possible to detect that it is of the form $A^{\cdot}$ for some object A of $\mathcal{C}$.

To do so we must characterise natural transformations for such functors.

Elements of $F(A)$ as natural transformations

Given a morphism $x : X \to A$ in $\mathcal{C}$, the functoriality of F gives a set map $F(x) : F(A) \to F(X)$.

Thus, for every element α of $F(A)$ we get an element $F(x)(\alpha)$ of $F(X)$. In other words, we have a set map

$$F(A) \times A^{\odot}(X) \to F(X) \text{ given by } (\alpha, x) \mapsto F(x)(\alpha)$$

Given an element α of $F(A)$ and an element x of $A^{\cdot}(X)$, we get an element $F(x)(\alpha)$ of $F(X)$.

We can thus define a set map $\alpha_X : A^{\cdot}(X) \to F(X)$ by $\alpha_X(x) = F(x)(\alpha)$.

(Note that a set map $X \times Y \to Z$, can be thought of as a map from $X \to \text{Map}(Y, Z)$ *or* a map from $Y \to \text{Map}(X, Z)$.)

Given a morphism $f : Y \to X$ in $\mathcal{C}$, we note that $F(x \circ f) = F(f) \circ F(x)$ since

F is contravariant. This gives

$$
\begin{aligned}
\left(\alpha_Y \circ A^{\cdot}(f)\right)(x) &= \alpha_Y \left(A^{\cdot}(f)(x)\right) \\
&= \alpha_Y(x \circ f) = F(x \circ f)(\alpha) \\
&= \left(F(f) \circ F(x)\right)(\alpha) = F(f)\left(F(x)(\alpha)\right) \\
&= F(f)\left(\alpha_X(x)\right) = \left(F(f) \circ \alpha_X\right)(x)
\end{aligned}
$$

This shows that the following diagram commutes

$$
\begin{CD}
A^{\cdot}(X) @>{\alpha_X}>> F(X) \\
@VV{A^{\cdot}(f)}V @VV{F(f)}V \\
A^{\cdot}(Y) @>{\alpha_Y}>> F(Y)
\end{CD}
$$

Thus, we have shown that an element $\alpha \in F(A)$ gives a natural transformation $A^{\cdot} \to F$ of functors $\mathcal{C}^{\mathrm{opp}}$ to **Set**.

Natural Transformations as elements of $F(A)$

For a natural transformation $\eta : A^{\cdot} \to F$ we have:

- For each object X of $\mathcal{C}$ a set map $\eta_A : A^{\cdot}(X) \to F(X)$.
- For each morphism $f : Y \to X$ of $\mathcal{C}$ a commutative diagram

$$
\begin{CD}
A^{\cdot}(X) @>{\eta_X}>> F(X) \\
@VV{A^{\cdot}(f)}V @VV{F(f)}V \\
A^{\cdot}(Y) @>{\eta_Y}>> F(Y)
\end{CD}
$$

Let us define α as the image of the identity map 1_A as an element of $A^{\cdot}(A) = \mathcal{C}(A, A)$.

Given an object Y of $\mathcal{Y}$ and an element of $y \in A^{\cdot}(Y) = \mathcal{C}(X, A)$, we apply the above diagram to the morphism $y : Y \to A$ to get

$$
\begin{CD}
A^{\cdot}(A) @>{\eta_X}>> F(A) \\
@VV{A^{\cdot}(y)}V @VV{F(y)}V \\
A^{\cdot}(Y) @>{\eta_Y}>> F(Y)
\end{CD}
$$

Let us calculate the image of 1_A in $A^{\cdot}(A)$ under both sides. We note that its image under the top arrow is α by definition. Hence, via one route we get $F(y)(\alpha)$.

On the other hand, the image of 1_A under $A^{\cdot}(y)$ is $1_A \circ y = y$. Thus, via the other route, we get $\eta_Y(y)$.

By comparing with the definition of α_Y given in the previous subsection, we get $\eta_Y(y) = \alpha_Y(y)$. Since this is for an *arbitrary* choice of Y and y, we see that $\eta_Y = \alpha_Y$.

Combining these two sections we obtain a natural identification between elements of $F(A)$ and natural transformations $A^{\cdot} \to F$. This is the statement of Yoneda Lemma.

Definition

Given a functor F from $\mathcal{C}^{\text{opp}}$ to **Set**, an object A of $\mathcal{C}$ and an element α in $F(A)$, we say that (A, α) *represents* the functor F if the natural transformation $\alpha : A^{\cdot} \to F$ as given by Yoneda Lemma is an isomorphism of functors.

Since, for each object X in $\mathcal{C}$, we have a set map $\alpha_X : A^{\cdot}(X) \to F(X)$, this is the same as asking that this set map is a bijection of sets.