Quiz 2: Categories of Functors

Question: Given the categories $\mathcal{C}$ and $\mathcal{D}$ as follows, describe the category $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ of functors.

The category $\mathcal{C}$ has one object and morphisms are given by non-negative integers with composition as addition.

The category $\mathcal{D}$ has one object and morphisms are given by positive integers with composition as multiplication.

Answer: We first compute the functors from $\mathcal{C}$ to $\mathcal{D}$. Each such functor takes the object to itself and takes the morphism 0 in $\mathcal{C}$ to the morphism 1 in $\mathcal{D}$ (since a functor takes the identity to the identity). Suppose the morphism 1 in $\mathcal{C}$ is taken to the morphism k in $\mathcal{D}$ for some positive integer k. Then, one proves by induction that the morphism n of $\mathcal{C}$ goes to the morphism k^n in $\mathcal{D}$.

Thus the objects in $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ can be identified with positive integers k where the functor corresponding to k takes the morphism n in $\mathcal{C}$ to the morphism k^n in $\mathcal{D}$. Let us denote the functor associate with k as F_k so that $F_k(n) = k^n$.

A natural transformation $r_{k,k'} : F_k \to F_{k'}$ associates to the unique object of $\mathcal{D}$ a morphism r in $\mathcal{D}$ which satifies a certain commutative diagram. This diagram says for every morphism n in $\mathcal{C}$, we have $r \circ F_k(n) = F_{k'}(n) \circ r$. This gives the identity $r \cdot k^n = k'^n \cdot r$ for every non-negative integer n.

There are two cases to consider. When $k = k'$ we get this identity for every positive integer r.

When $k \neq k'$, then the above identity *cannot* hold for $n = 1$ and a positive integer r.

Thus, the only morphisms are $r_{k,k} : F_k \to F_k$ for every positive integer r.

In other words, the category consists of one object F_k for each positive integer k, and one morphism $r_{k,k} : F_k \to F_k$ for each pair (r, k) of positive integers.