Adjoint functors

Adjoint functors occur in many places. We will see the definitions and some examples.

Concepts

A functor F from $\mathcal{C}$ to $\mathcal{D}$ is said to be a left adjoint of a functor G from $\mathcal{D}$ to $\mathcal{C}$ (equivalently, G is said to be a right adjoint of F), if there is a "natural" identification, for every object A of $\mathcal{C}$ and every object B of $\mathcal{D}$, between $\mathcal{C}(A, G(B))$ and $\mathcal{D}(F(A), B)$.

The question of what we mean by "natural" identification remains!

Note that such an identification can, in particular, be applied in the case $B = F(A)$. This gives an identification between $\mathcal{D}(F(A), F(A))$ and $\mathcal{C}(A, GF(A))$, where GF denotes the composite functor from $\mathcal{C}$ to itself. Now, $\mathcal{D}(F(A), F(A))$ contains a special element $1_{F(A)}$ and this suggests that there is a special element $u_A : A \to GF(A)$.

In other words, we should expect a natural transformation u from the identity functor on $\mathcal{C}$ to the functor GF.

Given such a natural transformation, and a morphism $g : F(A) \to B$, we can form the composite morphism

$$\overbrace{A \xrightarrow{u_A} GF(A) \xrightarrow{G(g)} G(B)}^{G(g)\circ u_A}$$

It would thus seem that we should define $g^u = G(g) \circ u_A$ so that

$$\mathcal{D}(F(A), B) \to \mathcal{C}(A, G(B)) \text{ given by } g \mapsto g^u$$

should give the required identification. Note that so far we have not shown that this map of sets is one-to-one and onto.

If we expect it to be onto, then for each object B and the case $A = G(B)$, there should be a morphism v_B in $\mathcal{C}(FG(B), B)$ such that $1_{G(B)} = (v_B)^u$. In other words, we should have a natural transformation $v_B : FG(B) \to B$ such that

$$1_{G(B)} = G(v_B) \circ u_{G(B)}$$

which can be seen as a commutative diagram of natural transformations

$$\begin{array}{ccc} G(B) & \xrightarrow{u_{G(B)}} & GFG(B) \\ & {}_{1_{G(B)}}\searrow\!\!\!\!\searrow & \downarrow{\scriptstyle G(v_B)} \\ & & G(B) \end{array}$$

Given a morphism $f : A \to G(B)$, we can now form the composite morphism

$$F(A) \xrightarrow{F(f)} FG(B) \xrightarrow{v_B} B \qquad \left(F(A) \xrightarrow{v_B \circ F(f)} B \right)$$

and so, we have

$$\mathcal{C}(A, G(B)) \to \mathcal{C}(F(A), B) \text{ given by } f \mapsto f_v = v_B \circ F(f)$$

We want this map to be the *inverse* of the map given above in order to complete the natural identification.

Note that $1^u_{F(A)} = G(1_{F(A)}) \circ u_A = 1_{GF(A)} \circ u_A = u_A$. Thus, we require the transposed identity

$$1_{F(A)} = (u_A)_v = v_{F(A)} \circ F(u_A)$$

This can be seen as a commutative diagram of natural transformations

$$\begin{array}{ccc} F(A) & \xrightarrow{F(u_A)} & FGF(A) \\ & {}_{1_{F(A)}} \searrow\!\!\searrow & \downarrow v_{F(A)} \\ & & F(A) \end{array}$$

Definition

A pair (F, G) of adjoint functors between categories $\mathcal{C}$ and $\mathcal{D}$ can be characterised as follows.

- There is a natural transformation $u : 1_{\mathcal{C}} \to GF$, called the *unit* of the adjunction.
- There is a natural transformation $v : FG \to 1_{\mathcal{D}}$, called the *counit* of the adjunction
- We have $1_F = v_F \circ F(u)$ and $1_G = G(v) \circ u_G$ corresponding the the above commutative diagrams.
- The maps $g \mapsto g^u = G(g) \circ u_A$ and its inverse $f \mapsto f_v = v_B \circ F(f)$ provide a natural identification between $\mathcal{D}(A, G(B))$ and $\mathcal{C}(G(A), B)$.

Free Monoid

Given a set S, we treat it as an "alphabet" and let S^* be the collection of all *finite* "strings" in this alphabet. In other words, elements of S^* can be identified as the union of the sets of n-tuples of elements of S for all $n \geq 0$. (Note that $n = 0$ represents the empty string.)

Strings can be concatenated. This operation is associative and the empty string plays the role of identity. Hence, we see that S^* is a monoid. It is often called the *free* monoid generated by S.

The assignment of S^* to S gives a functor F from **Set** to **Mon**, where the latter is the category of monoids. This is easily checked.

We also have the forgetful functor G from **Mon** to **Set** which associates to each monoid its underlying set and to a homomorphism of monoids associates the underlying set map.

Note that $GF(S) = S^*$ considered merely as a set rather than as a monoid. There is a natural set map $u_S : S \to S^*$ that takes an element $s \in S$ to the string consisting of just s.

Given a monoid M, we have a monoid $FG(M) = M^*$ which consists of strings of elements of M where the latter is considered just as a set. We have a natural map $M^* \to M$ which sends a string $(m_1, \ldots, m_k)$ to the product $m_1 \cdot \cdots \cdot m_k$ as elements of the monoid M. The associative law for M implies that this is a homomorphism of monoids. This gives a monoid homomorphism $v_M : FG(M) \to M$.

One checks easily that u and v are natural transformations and that the previous conditions are satisfied to make F the left adjoint functor of G.

Free Abelian Group

Given a set S, we treat it as a collection $\{e_s\}_{s\in S}$ of *formal* additive variables and let $\langle S\rangle = \oplus_{s\in S}\mathbb{Z}e_s$ be the collection of all *finite* "sums and differences" of these variables. In other words, elements of $\langle S\rangle$ can be identified as the collection of tuples of integers $(n_s)_{s\in S}$ indexed by S such that *all but finitely many* n_s *are* 0. It is sometimes convenient to represent such a tuple as a sum $\sum_f n_s e_s$ where the subscript f indicates that the sum is finite. Note that such tuples can be added "entry by entry" and this makes $\langle S\rangle$ into an Abelian group.

The assignment of $\langle S\rangle$ to S gives a functor F from **Set** to **Ab**, where the latter is the category of Abelian groups. This is easily checked.

We also have the forgetful functor G from **Ab** to **Set** which associates to each Abelian group its underlying set and to a homomorphism of Abelian groups associates the underlying set map.

Note that $GF(S) = \langle S\rangle$ considered merely as a set rather than as an Abelian group. There is a natural set map $u_S : S \to \langle S\rangle$ that takes an element $s \in S$ to the element where $n_s = 1$ and all other $n_{s'}$ for $s' \neq s$ are 0; equivalently, we take s to the variable e_s.

Given an Abelian group A, we have the Abelian group $FG(A) = \langle A\rangle$ where A is considered as a set. There is a natural map $\langle A\rangle \to A$ which sends a formal sum $\sum_f n_a e_a$ to the sum $\sum_f n_a a$ where we are treating A as an additive group. The associative law for A implies that this is a homomorphism of Abelian groups. This gives a Abelian group homomorphism $v_A : FG(A) \to A$.

One checks easily that u and v are natural transformations and that the previous conditions are satisfied to make F the left adjoint functor of G.

Polynomial Ring

Given a set S, we can treat it is a collection of variables $\{x_s\}_{s\in S}$, and let $\mathbb{Z}[S]$ denote the collection of polynomials in these variables. Under the usual rules of addition and multiplication of polynomials, we see that this is a commutative ring (with identity).

The assignment of $\mathbb{Z}[S]$ to S gives a functor F from **Set** to **CRing**, where the latter is the category of commutative rings with identity. This is easily checked.

We also have the forgetful functor G from **CRing** to **Set** which associates to each commutative ring its underlying set and to a homomorphism of commutative rings associates the underlying set map.

Note that $GF(S) = \mathbb{Z}[S]$ considered merely as a set rather than as a commutative ring. There is a natural set map $u_S : S \to \mathbb{Z}[S]$ that takes an element $s \in S$ to the variable $x_s = 1$.

Given an commutative ring with identity R, we have the commutative ring $FG(R) = \mathbb{Z}[R]$ where R is considered as a set. There is a natural ring homomorphism $\mathbb{Z}[R] \to R$ which sends the variable x_a to the element a in R; recall that a map from a polynomial ring to a commutative ring is *determined* by what it does to each variable. This gives a ring homomorphism $v_R : FG(R) \to R$.

One checks easily that u and v are natural transformations and that the previous conditions are satisfied to make F the left adjoint functor of G.

Abelianisation

Kiven a group K, we can form the Abelian group $K^{\text{ab}} = K/[K, K]$ where $[K, K]$ denotes the normal subgroup of K generated by elements of the form $[a, b] = a^{-1}b^{-1}ab$.

Note that if $K \to H$ is a group homomorphism, then the image of $[K, K]$ is contained in $[H, H]$. Thus, we see that we have an induced group homomorphism $K^{\text{ab}} \to H^{\text{ab}}$. Thus, the assignment of K^{a} to the group K gives a functor from **Grp** to **Ab**.

We also have a natural "forgetful" functor G from **Ab** to **Grp** that treats an Abelian group as just a group and "forgets" that it is an Abelian group.

Note that $GF(K) = K^{\text{ab}}$ considered as a group rather than as an Abelian group. The quotient homomorphism $K \to K/[K, K] = K^{\text{ab}}$ gives a natural transformation $u_K : K \to GF(K)$.

Given an Abelian group A, we see that $[A, A]$ is the trivial group. It follows that $FG(A) = A$ and thus, the identity map $v_A : FG(A) \to A$ gives the required natural transformation.

One checks easily that u and v are natural transformations and that the previous conditions are satisfied to make F the left adjoint functor of G.

Discrete/Indiscrete topology

So far, all our examples have been of an algebraic kind. Consider the functor G from **Top** to **Set** that associates to a topological space the underyling set.

If S is a set, then a set map $S \to G(X)$ is *automatically* continuous as a map $S \to X$ if we give S the discrete topology. This suggests that we define the left-adjoint to G as the functor F from **Set** to **Top** that associates to a set S, the topological space S_d which has the same underlying set with the discrete topology. As seen above, $\mathrm{Map}(S, G(X))$ can be identified with $\mathrm{Cont}(S_d = F(S), X)$. This easily leads to the required adjunction.

If S is a set, then a set map $G(X) \to S$ is *automatically* continuous as a map $X \to S$ if we give S the indiscrete topology; this is the topology where the only two open sets are the empty set and the whole space. This suggests that we define a right-adjoint to G as a functor H from **Set** to **Top** that associates to a set S, the topological space S_i which has the same underlying set with the indiscrete topology. As seen above, $\mathrm{Map}(G(X), S)$ can be identified with $\mathrm{Cont}(X, S_i)$ which leads to the required adjunction.

Compact-open topology

Given a locally-compact Hausdorff topological space X, we define a topology on $\mathrm{Cont}(X, Y)$ by prescribing the basic open sets $S(K, U)$ for $K \subset X$ compact and $U \subset Y$ open, defined by

$$S(K, U) = \{f : X \to Y | f \text{ continuous and } k(K) \subset Y\}$$

One can then prove the theorem that $X \times Z \to Y$ is continuous if and only if the resulting map $Z \to \mathrm{Cont}(X, Y)$ is continuous with this topology.

We define the functor $F = F_X$ from **Top** to itself as the one that sends a space Z to the space $X \times Z$.

We define the functor $G = G_X$ from **Top** to itself as the one that sends a space Y to the space $\mathrm{Cont}(X, Y)$ with the compact-open topology.

The above theorem can then be used to show that (F, G) is an adjoint pair.