Isomorphisms and Equivalences

Now that we have defined categories and functors between them, it makes sense to look at the question of isomorphisms.

Isomorphisms

A functor F from a category $\mathcal{C}$ to a category $\mathcal{D}$ can be said to be *surjective* if every object of $\mathcal{D}$ is of the form $F(A)$ for some object A in $\mathcal{C}$, and every morphism in $\mathcal{D}$ is of the form $F(f)$ for some morphism f in $\mathcal{C}$.

If we *only* have the condition that every morphism $F(A)$ to $F(B)$ is of the form $F(f)$ (and not the condition that every object is of the form $F(A)$), we say that the functor F is *full*.

A functor F from a category $\mathcal{C}$ to a category $\mathcal{D}$ can be said to be *injective* if distinct objects of $\mathcal{C}$ go to distinct objects in $\mathcal{D}$, and distinct morphisms in $\mathcal{D}$ go to distinct morphisms in $\mathcal{C}$.

If we *only* have the condition that $F(f) = F(g) : F(A) \to F(B)$ implies $f = g : A \to B$ (and not the condition that every $F(A) = F(B)$ implies $A = B$), we say that the functor F is *faithful*.

A functor F from a category $\mathcal{C}$ to a category $\mathcal{D}$ can be said to be an isomorphism if it is injective and surjective. Clearly, this is the same as saying that there is *exactly* one object of $\mathcal{C}$ for every chosen object of $\mathcal{D}$, and *exactly* one morphism of $\mathcal{C}$ for every chosen morphism of $\mathcal{D}$.

It follows that we can equivalently require that there is a functor G from $\mathcal{D}$ to $\mathcal{C}$ such that the composite functors $G \circ F$ and $F \circ G$ are identity functors on $\mathcal{C}$ and $\mathcal{D}$ respectively.

The functor F *identifies* objects and morphisms of $\mathcal{C}$ with those of $\mathcal{D}$. Hence, in some ways this is *not* particularly interesting!

Equivalences

We can weaken the above requirements as follows. As before we require a functor F from $\mathcal{C}$ to $\mathcal{D}$ and a functor G from $\mathcal{D}$ to $\mathcal{C}$. Instead of requiring $G \circ F$ to *be* the identity functor, we only require that there be a natural transformation $\eta : G \circ F \to 1_{\mathcal{C}}$ which is an isomorphism. Similarly, we require a natural transformation $\tau : F \circ G \to 1_{\mathcal{D}}$ which is an isomorphism. If such functors exist, we say that $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* categories.

Finite Sets

We use **Fin** to denote the category of finite sets. Note that even though the sets are finite, the objects of this category do not form a set! For every set S, the set $\{S\}$ is an object in **Fin**. Thus, we have a subcollection of the objects of **Fin** which has a one-to-one correspondence with the collection of all sets.

We use $\mathbb{F}$ to denote that category whose objects are counting numbers $0, 1, \ldots,$. Each object n of $\mathbb{F}$ can also be thought of as the finite set $[1, n]$ of natural numbers from 1 to n. Note that this set is empty for $n = 0$ and in general has exactly n elements. The morphisms from n to m are identified with set maps from $[1, n]$ to $[1, m]$.

Clearly, there is a functor A from $\mathbb{F}$ to **Fin** which takes n to the set $[1, n]$.

For each finite set S, the cardinality $n = |S|$ of S is a counting number. We *choose* one bijection (one-to-one onto map) $f_S : S \to [1, n]$. We define the functor B from **Fin** to $\mathbb{F}$ which sends a finite set S to its cardinality $B(S) = |S|$, and a set map $g : S \to T$ to the morphism $B(f) = f_T \circ g \circ f_S^{-1}$. One checks easily that B is a functor.

Note that $B \circ A$ is identity on objects of $\mathbb{F}$ since $|[1, n]| = n$. However, it need *not* be the identity on morphisms since we may not have chosen $f_{[1,n]} : [1, n] \to [1, n]$ to be the identity map! However, note that $f_{[1,n]}$ *is* an isomorphism. Even if we choose $f_{[1,n]}$ to be the identity map, we will see below that $A \circ B$ is *not* identity! In any case, we check that $\eta_n = f_{[1,n]}$ gives a natural transformation from $B \circ A$ to $1_{\mathbb{F}}$.

Given an object S of **Fin**, we see that $A \circ B(S) = [1, |S|]$ which need not be the set S when S is not the set $[1, n]$. However, in order to define B we have chosen an isomorphism $f_S : S \to [1, |S|]$. Thus, we have a natural transformation defined by $\tau_S = f_S^{-1}$ from $A \circ B$ to $1_{\mathbf{Fin}}$ as can be easily checked.

Note that the objects and morphisms of $\mathbb{F}$ form sets. This is an example of a "small" category which we shall introduce shortly. The point is that **Fin** is *not* a small category. However, it is equivalent to one. Since many behaviours of categories are preserved under equivalence, studying $\mathbb{F}$ is "enough" in order to study **Fin**.

Finite dimensional linear spaces

Fix a field k. We will be looking at finite dimensional linear spaces (vector spaces) over k.

We use $\mathbf{FLin}_k$ to denote the category of finite dimensional linear spaces over the field k with morphisms as k-linear maps.

We use $\mathbb{M}_k$ to denote that category whose objects are counting numbers $0, 1, \ldots,$. Each object n of $\mathbb{M}_k$ can also be thought of as the finite dimensional k-linear space k^n. Note that this is the vector space $\{0\}$ for $n = 0$. The morphisms from n to m are identified with $m \times n$ matrices and matrix multiplication gives composition. We identify $0 \times n$ matrices and $n \times 0$ matrices with the single element 0 which gives a unique map $n \to 0$, respectively $0 \to n$.

Clearly, there is a functor A from $\mathbb{M}_k$ to $\mathbf{FLin}_k$ which takes n to the linear space k^n where we think of this space as consisting of *column* vectors and matrix multiplication as giving the linear map $A(f) : k^n \to k^m$ associated with a matrix f.

For each finite dimensional linear space L, the dimension $n = \dim(L)$ of L is a counting number. We *choose* one k-linear isomorphism $v_L : k^n \to L$. This corresponds to a choice $(v_1, \ldots, v_n)$ of basis of L with map defined by $v_L((a_1, \ldots, a_n)) = \sum_{i=1}^n a_n v_n$.

We define $B(L) = n$ for an n-dimensional linear space. A linear map $g : L \to M$ is associated to the matrix $B(g)$ of the map $v_M^{-1} \circ g \circ v_L$ from k^n to k^m (with respect to the standard basesis. This is the *same* as taking the matrix associated to the linear g with respect to the chosen bases of L and M.

Note that $B \circ A$ is identity on objects of $\mathbb{M}_k$ since the dimension of k^n is n. Note that it need *not* be the identity on morphisms since we may not have chosen the standard basis for k^n! However, note that v_{k^n} *is* an isomorphism. Even if we choose the standard basis for k^n, we will see below that $A \circ B$ is *not* identity! In any case, we check that $\eta_n = v_{k^n}$ gives a natural transformation from $B \circ A$ to $1_{\mathbb{M}_k}$.

Given an object L of $\mathbf{FLin}_k$, we see that $A \circ B(L) = k^n$ which need not be the space L when L is not the space k^n. In order to define B we have chosen an isomorphism $v_L : L \to k^{\dim(S)}$. Thus, we have a natural transformation defined by $\tau_L = v_L$ from $A \circ B$ to $1_{\mathbf{FLin}_k}$ as can be easily checked.

Again note that the objects and morphisms of $\mathbb{M}_k$ form sets. Hence, this is also an example of a "small" category. The point is that $\mathbf{FLin}_k$ is *not* a small category. What we have shown is that it is equivalent to one. Thus, studying $\mathbb{M}_k$ is "enough" in order to study $\mathbf{FLin}_k$.