End-Semester Examination

MTH437 — Introduction to Schemes

- All questions carry equal marks.
- Use a separate page for the answer to each question.
- One page should be enough for each answer. Do *not* write long-winded answers!
- Please submit answers as PDF files containing *all* pages
- The answers *must* be submitted before midnight on 4th December 2021.

Q1. What is the number of $\mathbb{F}_2$ points in the quasi-projective variety $P(x_0, x_1, x_2, x_3; x_0x_3 - x_1x_2;)$? Generalise this to $\mathbb{F}_p$ and justify your answer.

Solution 1. We want 4-tuples (a_0, a_1, a_2, a_3) of elements of $\mathbb{F}_p$ such that not all are 0 and which satisfy $a_0a_3 = a_1a_2$.

$a_0 \neq 0$**:** Then $a_3 = a_1a_2/a_0$ is determined by a_1 and a_2. This gives $p-1$ values for a_0 and p values for each of a_1 and a_2. Thus, $p^2(p-1)$ tuples.

$a_0 = 0$**:** Then $a_1a_2 = 0$ which means at least one of them is 0.

$a_0 = 0$ **and** $a_1 = 0$**:** Then a_2 and a_3 cannot both be 0. So there are $p^2 - 1$ possibilities for their values.

$a_0 = 0$ **and** $a_1 \neq 0$**:** When $a_1 \neq 0$, then $a_2 = 0$ and a_3 can be anything. There are $(p-1)p$ possibilities for the values of a_1 and a_3.

Thus, there are, in total $p^2(p-1) + (p^2-1) + (p-1)p$ possible 4-tuples.

Two 4-tuples which are unit multiples of each other give the *same* point. So we need to divide this by $p-1$ to get

$$\frac{p^2(p-1) + (p^2-1) + (p-1)p}{p-1} = p^2 + p + 1 + p = (p+1)^2$$

Note that this can also be seen by noting that this variety is the Segre embedding which exhibits $\mathbb{P}^1 \times \mathbb{P}^1$ as a closed subfunctor of $\mathbb{P}^3$. Since we have checked that $\mathbb{P}^1(\mathbb{F}_p)$ has $(p+1)$ elements, the result follows!

Q2. What is the number of $\mathbb{F}_2$ points in the quasi-projective variety $P(x_0, x_1, x_2, x_3; \ ; x_0x_3, x_1)$? Generalise this to $\mathbb{F}_p$ and justify your answer.

Solution 2. We want 4-tuples (a_0, a_1, a_2, a_3) of elements of $\mathbb{F}_p$ such that not all are 0 such that either a_1 is non-zero or a_0a_3 is non-zero. We also need to equate two 4-tuples which are unit multiples of each other.

$a_1 \neq 0$**:** Upto equivalence we can take $a_1 = 1$. We can then take *any* values of a_0, a_2 and a_3. Thus, there are p^3 possibilities up to equivalence.

$a_1 = 0$ **and** $a_0a_3 \neq 0$**:** We can take *any* values of a_2 and only non-zero values of a_0 and a_3. Upto equivalence, we can take $a_0 = 1$. Then there are $p(p-1)$ possibilities for the pair (a_2, a_3).

We thus see that there are $p^3 + p^2 - p$ possibilities.

We can also see this as the *complement* in $\mathbb{P}^3$ of the projective variety $P(x_0, x_1, x_2, x_3; x_0x_3, x_1)$. This can be seen as the join of two $\mathbb{P}^1(\mathbb{F}_p)$'s along a common point; this has $2(p+1) - 1$ points. Since $\mathbb{P}^3$ has $(p^4 - 1)/(p-1)$ points we get

$$\frac{p^4 - 1}{p - 1} - 2(p+1) + 1 = p^3 + p^2 + p + 1 - 2p - 1 = p^3 + p^2 - p$$

Q3. Given the affine schemes $X = A(x; x^2 + 5)$ and $Y = A(x, y, z; xy + z^2 - 1)$. Produce a morphism exhibiting X as a subfunctor of Y. Justify your answer.

Solution 3. This is a morphism of affine schemes and thus corresponds to a ring homomorphism

$$\mathbb{Z}[x, y, z]/\langle xy + z^2 - 1\rangle \to \mathbb{Z}[x]/\langle x^2 + 5\rangle = \mathbb{Z}[\sqrt{-5}]$$

We note that $2 \cdot 3 + (\sqrt{-5})^2 = 1$ in the latter ring.

Thus, we can take the homomorphism given by $(x, y, z) \mapsto (2, 3, \sqrt{-5})$.

Equivalently, we can see this as a $\mathbb{Z}[\sqrt{-5}]$-point of the affine scheme $Y = A(x, y, z; xy + z^2 - 1)$.

Note that the ring homomorphism is *onto*. This shows that X is a closed subfunctor of Y under this morphism.

Q4. Show that the following give points of $\mathbb{P}^1$:

- $(\sqrt{-5} - 1 : 2)$ is an $\mathbb{Z}[\sqrt{-5}, 1/2]$-point
- $(-3 : \sqrt{-5} + 1)$ is an $\mathbb{Z}[\sqrt{-5}, 1/3]$-point

Show that the two points *restrict* to the same $\mathbb{Z}[\sqrt{-5}, 1/6]$-point of $\mathbb{P}^1$ under the natural homomorphisms $\mathbb{Z}[\sqrt{-5}, 1/2] \to \mathbb{Z}[\sqrt{-5}, 1/6]$ and $\mathbb{Z}[\sqrt{-5}, 1/3] \to \mathbb{Z}[\sqrt{-5}, 1/6]$.

Justify that there is an $\mathbb{Z}[\sqrt{-5}]$-point of $\mathbb{P}^1$ that restics to these points under the natural homomorphisms $\mathbb{Z}[\sqrt{-5}] \to \mathbb{Z}[\sqrt{-5}, 1/2]$ and $\mathbb{Z}[\sqrt{-5}] \to \mathbb{Z}[\sqrt{-5}, 1/3]$.

Solution 4. Note that 2 is a unit in $\mathbb{Z}[\sqrt{-5}, 1/2]$. This shows the first statement.

Note that 3 is a unit in $\mathbb{Z}[\sqrt{-5}, 1/3]$. This shows the second statement.

We note that in $R = \mathbb{Z}[\sqrt{-5}, 1/6]$ we have

$$-3(\sqrt{-5}-1, 2) = (-3\sqrt{-5}-1, -6) = (-3\sqrt{-5}-1, (\sqrt{-5}+1)(\sqrt{-5}-1)) = (\sqrt{-5}-1)\cdot(-3, \sqrt{-5}+1))$$

Moreover, 3 and $\sqrt{-5} - 1$ are both *units* in R. Thus, the tuples $(\sqrt{-5} - 1 : 2)$ and $(-3, \sqrt{-5} + 1)$ represent the same R-point of $\mathbb{P}^1$.

Note that:

- $\mathbb{P}^1$ is a sheaf functor
- 2 and 3 generate the unit ideal in $S = \mathbb{Z}[\sqrt{-5}]$.

Hence, by *patching* we obtain an S-point of of $\mathbb{P}^1$.

Q5. In which direction(s) are there morphisms between the following pairs of schemes? Justify your answers in each case.

1. $\mathrm{Sp}(\{0\})$ and $\mathbb{A}^1$.
2. $\mathrm{Sp}(\mathbb{Z})$ and $\mathrm{Sp}(\mathbb{Z}[1/2])$.
3. $\mathrm{Sp}(\mathbb{Z}[1/2])$ and $\mathrm{Sp}(\mathbb{F}_2)$.

Solution 5.

1. There is a unique morphism *from* the empty scheme $\mathrm{Sp}(\{0\})$ to any scheme. There are *no* morphisms *to* the empty scheme, except from itself!
2. There is a unique morphism from any scheme to $\mathrm{Sp}(\mathbb{Z})$. On the other hand there is no ring homomorphism $\mathbb{Z}[1/2] \to \mathbb{Z}$ since $1 + 1 = 2$ is a unit in the left-hand side.
3. There are no homomorphisms from either of these rings to the other ring since $2 = 0$ in one ring and 2 is a unit in the other ring.

Q6. Exhibit a morphism from $X = \mathbb{A}^2$ to $Y = P(x_0, x_1, x_2, x_3; x_0x_3 - x_1x_2)$ making X a subscheme of Y.

Solution 6. The open subscheme of Y where x_0 is a unit is given by the affine scheme $U = A(x_1, x_2, x_2; x_3 - x_1x_2)$. Clearly, we have a morphism $\mathbb{A}^2 = A(x, y;)$ to U given by

$$(X, y) \mapsto (x_1, x_2, x_3) = (x, y, xy)$$

Q7. Show that the map $z \mapsto z^2$ from $\mathbb{C}^*$ to itself is a local homeomorphisms with the usual topology. Hence, it gives a sheaf F over $\mathbb{C}^*$ in the classical sense.

What can be said about $F(\mathbb{C}^*)$?

Solution Q7. The function $f(z) = z^2$ has derivative $f'(z) = 2z \neq 0$ for z in $\mathbb{C}^*$. By the inverse function theorem it follows that it has a local *holomorphic* inverse. In particular, it has a local *continuous* inverse. Thus, it is a local homeomorphism.

Note that there is *no* global inverse function to f. Thus, $F(\mathbb{C}^*)$ is *empty*.

Q8. Does the morphism $x \mapsto x^2$ from $\mathrm{Spec}(\mathbb{C}[x, x^{-1}])$ to itself give a local homeomorphism in the Zariski topology? Justify your answer.

Solution Q8. This is *not* a local homeomorphism.

A proper closed subset of $\mathrm{Spec}(\mathbb{C}[x, x^{-1}])$ is a finite subset of $\mathbb{C}^*$. The complement of such a set *always* contains two points of the form $\{\pm a\}$ for some complex number a. In particular, the map restricted to the complement is *not* one-to-one. Hence, it cannot be a homeomorphism restricted to on *any* open set in this space.

Q9. Write the following affine scheme X as a *disjoint* union of two affine schemes

$$X = A(x_1, x_2, x_3; x_1x_2, x_3(x_3 - 1), x_1(x_3 - 1), x_2x_3)$$

Solution Q9. We note that x_3 is an idempotent in the co-ordinate ring $\mathcal{O}(X)$.

Thus we have two *disjoint* closed subschemes corresponding to putting $x_3 = 0$ and $x_3 = 1$.

$$X_1 = A(x_1, x_2, x_3; x_3 - 1, x_2)$$
$$X_2 = A(x_1, x_2, x_3; x_3, x_1))$$

such that X is the union of these two closed subschemes.

Note that X_1 and X_2 are "skew" lines (embedded $\mathbb{A}^1$'s) in the affine space $\mathbb{A}^3 = A(x_1, x_2, x_3)$.

Q10. Consider the following functors **CRing** to **Set**. Which of these are "sheaf functors" (i.e. satisfy the (co-)sheaf property)?

1. The functor U that associates to each commutative ring R the set $U(R)$ of units in R.
2. The functor N that associates to each commutative ring R the set $N(R)$ of nilpotent elements in R.

Solution Q10.

1. Note that $\mathrm{Hom}(\mathbb{Z}[x, x^{-1}] \to R$ can be identified with $U(R)$ by $f \mapsto f(x)$. This shows that U is the functor associated with the scheme $\mathrm{Sp}(\mathbb{Z}[x, x^{-1}]$. It follows that it is also a sheaf.

2. Given a ring R, $u_1, \ldots, u_k$ in R which generate the unit ideal and $x_i \in N(R_{u_i})$ that satisfy the patching condition we have to find x in $N(R)$ that restricts to x_i for each i.

   Note that $N(R) \subset R$ and $R = \mathbb{A}^1(R)$ is a sheaf. Thus, there is a unique x in R such that x restricts to x_i for each i. We only need to check whether x is nilpotent.

   Let n_i be such that $x_i^{n_i} = 0$ and n be the maximum of the n_i. We note that $y = x^n$ is an element of R which restricts to $x_i^n = 0$ for each i. By unique-ness of the patching for the sheaf $\mathbb{A}^1$, we see that $y = 0$ as required.