A more “geometric” definition

MTH437 — Introduction to Schemes

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Recall

For a scheme S, we introduced the category $\mathbf{Schemes}_S$ of schemes with a *chosen* morphism to S and morphisms that commute with this choice.

The objects are called S-schemes. When $S = \mathrm{Sp}(k)$ where k is a commutative ring, we also call them k-schemes.

We then introduced the full subcategory $\mathbf{FTScheme}_k$ of k-schemes of finite type.

The objects of this category are quotients of affine schemes $U = \mathrm{Sp}(R)$ where R is a *finitely generated* k-algebra.

The category $\mathbf{QProj}_k$ of quasi-projective schemes over k is a full subcategory of $\mathbf{FTScheme}_k$. This is the primary topic of interest in algebraic geometry.

Reduced Schemes

Given a commutative ring R, the collection N of all its nilpotent elements forms an ideal called the *nil radical* of R.

For any ring R we can form the quotient ring R/N which can be shown to have no nilpotent elements except 0; such a ring is called a *reduced* ring.

A scheme X which is the quotient (as usual) of $U = \sqcup_i U_i$ by a Zariski open equivalence relation $E = \sqcup_{i,j} V_{i,j}$, where $U_i = \mathrm{Sp}(R^{(i)})$ is called a *reduced* scheme if $R^{(i)}$ is a reduced ring for each i.

The full subcategory **RScheme** of **Scheme** consists of reduced schemes.

When k is reduced (e.g. k is a field) we can talk about $\mathbf{RFTScheme}_k$ and $\mathbf{RQProj}_k$ as well.

These reduced analogues of the categories of schemes are *not* well-behaved for categorical constructions.

For example, consider the morphism $\mathbb{A}^1$ to itself given by $x \mapsto x^n$.

The inverse image of the closed subscheme $A(x; x)$ is not reduced.

More generally, fibre-products need not be reduced. This causes various universal properties to become cumbersome to define and prove.

Zariski-Weil Foundations

Given a field k, let K be the algebraic closure of the field of quotients of the polynomial ring in infinitely many variables over k.

We consider K as a k-algebra in a natural way. This makes $\mathrm{Sp}(K)$ a k-scheme (which is far from being of finite type!).

To a k-scheme X of finite type, we associate the set $X(K)$ with open sets given by $U(K)$ where U is an open subscheme of X.

This defines a functor from $\mathbf{FTScheme}_k$ to topological spaces. We can further restrict this to $\mathbf{RFTScheme}_k$.

We can show that this functor is *faithful*. In other words, if $f, g : X \to Y$ are morphisms of reduced k-schemes such that $f(K) = g(K) : X(K) \to Y(K)$, then $f = g$.

We could try to study reduced k-schemes of finite type by studying the topological spaces $X(K)$ and a *suitable* subset of continuous maps $X(K) \to Y(K)$. This subset should somehow capture the idea that the map is "algebraic".

This is roughly the approach taken by Zariski, Weil and Chow.

The following problems arise:

- It is more problematic to deal with fibre-products. The latter arise in numerous geometric constructions like intersection theory.
- There does not seem to be a *nice* way to characterise the subset of continuous maps *unless* one further restricts the type of scheme that the domain X can be.

Finite k-algebras

We define $\mathbf{FA}_{\mathbb{Z}}$ to be the category of all *finite* commutative rings; i.e. rings where the underlying set is finite.

For a field k, we define $\mathbf{FA}_k$ to be the category of all *finite* k-algebras; i.e. k-algbras where the underlying k-vector space is finite dimensional.

Note that we can also think of such k-algebras as commuting k-algebras of matrices.

Now fix the ring k to be either $\mathbb{Z}$ or a field. We refer to $\mathbf{FA}_k$ as the category of finite k-algebras.

Note that in such an algebra A every chain of ideals is finite. In the first case, this is because there are only finitely many ideals. In the second case, this is because all ideals are k-subspaces.

It follows easily that:

- there are finitely many maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ in A.
- There exist non-negative integers $a_1, \ldots, a_r$ such that $\mathfrak{m}_1^{a_1} \cap \cdots \cap \mathfrak{m}_r^{a_r} = \{0\}$.

In particular A is naturally isomorphic to the finite direct sum of the rings $A_i = A/\mathfrak{m}_i^{a_i}$ which are *local* finite k-algebras.

The key idea is that for a k-scheme of finite type X the functor $\mathbf{FA}_k$ to **Set** which associates to A the set $X(A)$, is *sufficient* to determine X.

Lemma: Given morphisms $f, g : X \to Y$ of k-schemes of finite type such that $f(A) = g(A) : X(A) \to Y(A)$ for *every* A in $\mathbf{FA}_k$, we have $f = g$.

This follows from the following statement about finitely generated k-algebras.

Lemma: If a morphism $f : X \to Y$ of k-schemes of finite is such that $f(A) : X(A) \to Y(A)$ is a bijection for *every* A in $\mathbf{FA}_k$, then f is an isomorphism.

This can be *reduced* to the case $f : \mathrm{Sp}(R) \to \mathrm{Sp}(S)$ for finitely generated k-algebras and then proved using some standard techniques in commutative algebra.

Functors on $\mathbf{FA}_k$

Given a quasi-projective k-scheme

$$X = Q_k(x_0, \ldots, x_p; f_1, \ldots, f_q; g_1, \ldots, g_r)$$

We want to understand the associated functor from $\mathbf{FA}_k$ to **Set**.

We have seen that for a local ring A, the points of $X(A)$ are equivalence classes of the form $(a_0 : \cdots : a_p)$ where:

- $(a_0 : \cdots : a_p) = (ua_0 : \cdots : ua_p)$ for a unit u in A
- $f_j(\mathbf{a}) = 0$ for all j, and
- a_i is a unit for at least one i.
- $g_k(\mathbf{a})$ is a unit for at least one k.

Note that this is essentially how we would define points of the quasi-projective scheme over a field.

Using the expression of a finite k-algebra A as a finite direct sum of local rings, we can extend this to describe points of $X(A)$ as above with the following changes.

Note that the group of units in A is the *product* of the groups of units of each local ring.

For a finite k-algebra A, the points of $X(A)$ are equivalence classes of the form $(a_0 : \cdots : a_p)$ where:

- $(a_0 : \cdots : a_p) = (ua_0 : \cdots : ua_p)$ for a unit u in A
- $f_j(\mathbf{a}) = 0$ for all j, and
- $\langle a_0, \ldots, a_p \rangle = A$, and
- $\langle g_1(\mathbf{a}), \ldots, g_r(\mathbf{a}) \rangle = A$, and

Note that the target of the functor is also a very special (and small) collection of sets. These are subsets of $\mathbb{P}^p(A)$ as A varies over $\mathbf{FA}_k$.

Products

The scheme $\mathbb{P}^p \times \mathbb{P}^q$ can be identified as the subscheme of $\mathbb{P}^{pq+p+q}$ via the Segre embedding.

We write the coordinates on $\mathbb{P}^{pq+p+q}$ as $(z_{i,j})_{i=0;j=0}^{i=p,j=q}$ and consider the subscheme X defined by $z_{i,j}z_{k,l} = z_{i,l}z_{k,j}$.

For a finite k-algebra A one shows that the map $\mathbb{P}^p(A) \times \mathbb{P}^q(A) \to X(A)$ defined by

$$((a_0, \ldots, a_p), (b_0, \ldots, b_q),) \mapsto ((z_{i,j} = a_i b_j)_{i=0;j=0}^{i=p;j=q}$$

is a bijection.

This allows us to see that, if X and Y are quasi-projective schemes then so is $X \times Y$.

Correspondences

If $X \subset \mathbb{P}^p$ is a quasi-projective scheme, then a closed subscheme $Z \subset X$ is defined by the *additional* vanishing of some homogeneous polynomials.

A *correspondence* between two schemes X and Y is a closed subscheme of $X \times Y$.

We note that the *graph* of a morphism $f : X \to Y$ defines a closed subscheme $\Gamma_f \subset X \times Y$.

Note that the natural projection map $\Gamma_f \to X$ is an isomorphism of schemes.

By the lemma stated earlier this is *equivalent* to the assertion that for every finite k-algebra, the map $\Gamma_f(A) \to X(A)$ is a bijection.

Alternative definition of $\mathbf{QProj}_k$

For the "symbol" $Q(x_0, \ldots, x_p; f_1, \ldots, f_q; g_1, \ldots, g_r)$ we define the functor from $\mathbf{FA}_k$ to **Set** as above.

These are the objects of our category. Morphisms will be a *subset* of natural transformations as defined below.

The Segre embedding shows how, if X and Y are such objects then so is $X \times Y$.

We have defined above what it means for Y to be a closed subfunctor of X.

A morphism $f : X \to Y$ is a natural transformation such that its graph Γ_f is a closed subfunctor of $X \times Y$.

One might see this as more "geometric" definition of the category of quasi-projective schemes.