7.5. Show that if A, B are modules over a principal ideal domain and if $a \in A$, $b \in B$ are not torsion elements then $a \otimes b \neq 0$ in $A \otimes_\Lambda B$ and is not a torsion element.

7.6. Show that if A is a finitely generated module over a principal ideal domain and if $A \otimes_\Lambda A = 0$, then $A = 0$. Give an example of an abelian group $G \neq 0$ such that $G \otimes G = 0$.

7.7. Let $A \times B$ be the cartesian product of the sets underlying the right Λ-module A and the left Λ-module B. For G an abelian group call a function $f : A \times B \to G$ *bilinear* if

$$f(a_1 + a_2, b) = f(a_1, b) + f(a_2, b), \;\; a_1, a_2 \in A, \;\; b \in B;$$

$$f(a, b_1 + b_2) = f(a, b_1) + f(a, b_2), \;\; a \in A, \;\; b_1, b_2 \in B;$$

$$f(a\lambda, b) = f(a, \lambda b), \;\; a \in A, \;\; b \in B, \;\; \lambda \in \Lambda .$$

Show that the tensor product has the following universal property. To every abelian group G and to every bilinear map $f : A \times B \to G$ there exists a unique homomorphism of abelian groups

$$g : A \otimes_\Lambda B \to G \quad \text{such that} \quad f(a, b) = g(a \otimes b).$$

7.8. Show that an associative algebra (with unity) over the commutative ring Λ may be defined as follows. An algebra A is a Λ-module together with Λ-module homomorphisms $\mu : A \otimes_\Lambda A \to A$ and $\eta : \Lambda \to A$ such that the following diagrams are commutative

$$\begin{array}{ccccc} \Lambda \otimes_\Lambda A & \xrightarrow{\sim} & A & \xleftarrow{\sim} & A \otimes_\Lambda \Lambda \\ {\scriptstyle \eta \otimes 1}\downarrow & & \| & & \downarrow{\scriptstyle 1 \otimes \eta} \\ A \otimes_\Lambda A & \xrightarrow{\mu} & A & \xleftarrow{\mu} & A \otimes_\Lambda A \end{array} \qquad \begin{array}{ccc} A \otimes_\Lambda A \otimes_\Lambda A & \xrightarrow{\mu \otimes 1} & A \otimes_\Lambda A \\ {\scriptstyle 1 \otimes \mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ A \otimes_\Lambda A & \xrightarrow{\mu} & A . \end{array}$$

(The first diagram shows that $\eta(1_\Lambda)$ is a left and a right unity for A, while the second diagram yields associativity of the product.) Show that if A and B are algebras over Λ then $A \otimes_\Lambda B$ may naturally be made into an algebra over Λ.

7.9. An algebra A over Λ is called *augmented* if a homomorphism $\varepsilon : A \to \Lambda$ of algebras is given. Show that the group algebra KG is augmented with $\varepsilon : KG \to K$ defined by $\varepsilon(x) = 1$, $x \in G$. Give other examples of augmented algebras.

8. The Functor Tor

Let A be a right Λ-module and let B be a left Λ-module. Given a projective presentation $R \overset{\mu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} A$ of A we define

$$\operatorname{Tor}_\varepsilon^\Lambda(A, B) = \ker(\mu_* : R \otimes_\Lambda B \to P \otimes_\Lambda B).$$

The sequence

$$0 \to \operatorname{Tor}_\varepsilon^\Lambda(A, B) \to R \otimes_\Lambda B \to P \otimes_\Lambda B \to A \otimes_\Lambda B \to 0$$

is exact. Obviously we can make $\mathrm{Tor}_\varepsilon^\Lambda(A, -)$ into a covariant functor by defining, for a map $\beta : B \to B'$, the associated map

$$\beta_* : \mathrm{Tor}_\varepsilon^\Lambda(A, B) \to \mathrm{Tor}_\varepsilon^\Lambda(A, B')$$

to be the homomorphism induced by $\beta_* : R \otimes_\Lambda B \to R \otimes_\Lambda B'$. To any projective presentation $S \overset{\nu}{\rightarrowtail} Q \overset{\eta}{\twoheadrightarrow} B$ of B we define

$$\overline{\mathrm{Tor}}_\eta^\Lambda(A, B) = \ker(\nu_* : A \otimes_\Lambda S \to A \otimes_\Lambda Q) .$$

With this definition the sequence

$$0 \to \overline{\mathrm{Tor}}_\eta^\Lambda(A, B) \to A \otimes_\Lambda S \to A \otimes_\Lambda Q \to A \otimes_\Lambda B \to 0$$

is exact. Clearly, given a homomorphism $\alpha : A \to A'$, we can associate a homomorphism $\alpha_* : \overline{\mathrm{Tor}}_\eta^\Lambda(A, B) \to \overline{\mathrm{Tor}}_\eta^\Lambda(A', B)$, which is induced by $\alpha_* : A \otimes_\Lambda S \to A' \otimes_\Lambda S$. With this definition $\overline{\mathrm{Tor}}_\eta^\Lambda(-, B)$ is a covariant functor.

Proposition 8.1. *If A (or B) is projective, then*

$$\mathrm{Tor}_\varepsilon^\Lambda(A, B) = 0 = \overline{\mathrm{Tor}}_\eta^\Lambda(A, B) .$$

Proof. Since A is projective, the short exact sequence $R \overset{\mu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} A$ splits, i.e. there is $\kappa : P \to R$ with $\kappa\mu = 1_R$. Hence

$$\kappa\mu \otimes 1 = (\kappa \otimes 1)(\mu \otimes 1) = 1_{R \otimes_\Lambda B} ,$$

and consequently $\mu \otimes 1$ is monomorphic. Thus $\mathrm{Tor}_\varepsilon^\Lambda(A, B) = 0$.

If A is projective, A is flat by Proposition 7.4. Hence

$$0 \to A \otimes_\Lambda S \to A \otimes_\Lambda Q \to A \otimes_\Lambda B \to 0$$

is exact. Thus $\overline{\mathrm{Tor}}_\eta^\Lambda(A, B) = 0$. The remaining assertions merely interchange left and right. □

Next we will use Lemma 5.1 to show that $\overline{\mathrm{Tor}}_\eta^\Lambda$ and $\mathrm{Tor}_\varepsilon^\Lambda$ denote the same functor. Again let $R \overset{\mu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} A$ and $S \overset{\nu}{\rightarrowtail} Q \overset{\eta}{\twoheadrightarrow} B$ be projective presentations. We then construct the commutative diagram

$$\begin{array}{ccccccc}
 & & & & 0 & \longrightarrow & \mathrm{Tor}_\varepsilon^\Lambda(A, B) \\
 & & & & \downarrow & \Sigma_5 & \downarrow \\
 & & R \otimes_\Lambda S & \longrightarrow & R \otimes_\Lambda Q & \twoheadrightarrow & R \otimes_\Lambda B \\
 & & \downarrow & \Sigma_3 & \downarrow & \Sigma_4 & \downarrow \\
0 & \longrightarrow & P \otimes_\Lambda S & \longrightarrow & P \otimes_\Lambda Q & \twoheadrightarrow & P \otimes_\Lambda B \\
\downarrow & \Sigma_1 & \downarrow & \Sigma_2 & \downarrow & & \downarrow \\
\overline{\mathrm{Tor}}_\eta^\Lambda(A, B) & \longrightarrow & A \otimes_\Lambda S & \longrightarrow & A \otimes_\Lambda Q & \twoheadrightarrow & A \otimes_\Lambda B
\end{array} \qquad (8.1)$$

By a repeated application of Lemma 3.1 we obtain

$$\overline{\mathrm{Tor}}_\eta^\Lambda(A,B) = \mathrm{Im}\,\Sigma_1 \cong \mathrm{Ker}\,\Sigma_2 \cong \mathrm{Im}\,\Sigma_3 \cong \mathrm{Ker}\,\Sigma_4 \cong \mathrm{Im}\,\Sigma_5 = \mathrm{Tor}_\varepsilon^\Lambda(A,B)\,.$$

Now let $R' \overset{\mu'}{\rightarrowtail} P' \overset{\varepsilon'}{\twoheadrightarrow} A'$ be a projective presentation of A' and $\alpha: A \to A'$ a homomorphism. We can then find $\varphi: P \to P'$ and $\psi: R \to R'$ such that the following diagram commutes:

$$\begin{array}{ccccc} R & \overset{\mu}{\rightarrowtail} & P & \overset{\varepsilon}{\twoheadrightarrow} & A \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\alpha} \\ R' & \overset{\mu'}{\rightarrowtail} & P' & \overset{\varepsilon'}{\twoheadrightarrow} & A' \end{array} \tag{8.2}$$

These homomorphisms induce a map from the diagram (8.1) into the diagram corresponding to the presentation $R' \rightarrowtail P' \twoheadrightarrow A'$. Consequently we obtain a homomorphism

$$\mathrm{Tor}_\varepsilon^\Lambda(A,B) \xrightarrow{\sim} \overline{\mathrm{Tor}}_\eta^\Lambda(A,B) \xrightarrow{\alpha_*} \overline{\mathrm{Tor}}_\eta^\Lambda(A',B) \xrightarrow{\sim} \mathrm{Tor}_{\varepsilon'}^\Lambda(A',B)$$

which is visibly independent of the choice of φ in (8.2). Choosing $\alpha = 1_A$ we obtain an isomorphism $\mathrm{Tor}_\varepsilon^\Lambda(A,B) \xrightarrow{\sim} \overline{\mathrm{Tor}}_\eta^\Lambda(A,B) \xrightarrow{\sim} \mathrm{Tor}_{\varepsilon'}^\Lambda(A,B)$.

Collecting the information obtained, we have shown that there is a natural equivalence between the functors $\mathrm{Tor}_\varepsilon^\Lambda(A,-)$ and $\mathrm{Tor}_{\varepsilon'}^\Lambda(A,-)$, that we therefore can drop the subscript ε, writing $\mathrm{Tor}^\Lambda(A,-)$ from now on; further that $\mathrm{Tor}^\Lambda(-,B)$ can be made into a functor, which is equivalent to $\overline{\mathrm{Tor}}_\eta^\Lambda(-,B)$ for any η. We thus can use the notation $\mathrm{Tor}^\Lambda(A,B)$ for $\overline{\mathrm{Tor}}_\eta^\Lambda(A,B)$, also. We finally leave it to the reader to show that $\mathrm{Tor}^\Lambda(-,-)$ is a bifunctor. The fact that $\mathrm{Tor}^\Lambda(-,-)$ coincides with $\overline{\mathrm{Tor}}^\Lambda(-,-)$ is sometimes expressed by saying that Tor is *balanced*.

Similarly to Theorems 5.2 and 5.3, one obtains

Theorem 8.2. *Let A be a right Λ-module and $B' \overset{\kappa}{\rightarrowtail} B \overset{\nu}{\twoheadrightarrow} B''$ an exact sequence of left Λ-modules, then there exists a connecting homomorphism $\omega: \mathrm{Tor}^\Lambda(A,B'') \to A \otimes_\Lambda B'$ such that the following sequence is exact:*

$$\begin{aligned} \mathrm{Tor}^\Lambda(A,B') &\xrightarrow{\kappa_*} \mathrm{Tor}^\Lambda(A,B) \xrightarrow{\nu_*} \mathrm{Tor}^\Lambda(A,B'') \xrightarrow{\omega} A \otimes_\Lambda B' \\ &\xrightarrow{\kappa_*} A \otimes_\Lambda B \xrightarrow{\nu_*} A \otimes_\Lambda B'' \longrightarrow 0\,. \end{aligned} \tag{8.3}$$

Theorem 8.3. *Let B be a left Λ-module and let $A' \overset{\kappa}{\rightarrowtail} A \overset{\nu}{\twoheadrightarrow} A''$ be an exact sequence of right Λ-modules. Then there exists a connecting homomorphism $\omega: \mathrm{Tor}^\Lambda(A'',B) \to A' \otimes_\Lambda B$ such that the following sequence is exact:*

$$\begin{aligned} \mathrm{Tor}^\Lambda(A',B) &\xrightarrow{\kappa_*} \mathrm{Tor}^\Lambda(A,B) \xrightarrow{\nu_*} \mathrm{Tor}^\Lambda(A'',B) \xrightarrow{\omega} A' \otimes_\Lambda B \\ &\xrightarrow{\kappa_*} A \otimes_\Lambda B \xrightarrow{\nu_*} A'' \otimes_\Lambda B \longrightarrow 0\,. \end{aligned} \tag{8.4}$$

Proof. We only prove Theorem 8.2; the proof of Theorem 8.3 may be obtained by replacing Tor by $\overline{\mathrm{Tor}}$. Consider the projective presentation

$R \overset{\mu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} A$ and construct the diagram:

$$\begin{array}{ccccc}
\operatorname{Tor}^{\Lambda}(A,B') & \dashrightarrow & \operatorname{Tor}^{\Lambda}(A,B) & \dashrightarrow & \operatorname{Tor}^{\Lambda}(A,B) \\
\downarrow & & \downarrow & & \downarrow \\
R\otimes_{\Lambda}B' & \longrightarrow & R\otimes_{\Lambda}B & \longrightarrow & R\otimes_{\Lambda}B''\to 0 \\
\downarrow & & \downarrow & & \downarrow \\
0\to P\otimes_{\Lambda}B' & \longrightarrow & P\otimes_{\Lambda}B & \longrightarrow & P\otimes_{\Lambda}B''\to 0 \\
\downarrow & & \downarrow & & \downarrow \\
A\otimes_{\Lambda}B' & \dashrightarrow & A\otimes_{\Lambda}B & \dashrightarrow & A\otimes_{\Lambda}B'
\end{array} \qquad (8.5)$$

By applying Lemma 5.1 we obtain the asserted sequence. □

We remark that like the Hom-Ext sequences the sequences (8.3) and (8.4) are natural. Notice that by contrast with the two sequences involving Ext we obtain only *one* kind of sequence involving Tor, since A, B play symmetric roles in the definition of Tor.

Corollary 8.4. *Let Λ be a principal ideal domain. Then the homomorphisms $\kappa_* : \operatorname{Tor}^{\Lambda}(A, B') \to \operatorname{Tor}^{\Lambda}(A, B)$ in sequence* (8.3) *and*

$\kappa_* : \operatorname{Tor}^{\Lambda}(A', B) \to \operatorname{Tor}^{\Lambda}(A, B)$ *in sequence* (8.4) *are monomorphic.*

Proof. By Corollary I.5.3 R is a projective right Λ-module, hence the map $\kappa_* : R\otimes_{\Lambda} B' \to R\otimes_{\Lambda} B$ in diagram (8.5) is monomorphic, whence the first assertion. Analogously one obtains the second assertion. □

Exercises:

8.1. Show that, if A (or B) is flat, then $\operatorname{Tor}^{\Lambda}(A, B) = 0$.

8.2. Evaluate the exact sequences (8.3), (8.4) for the examples given in Exercise 5.7 (i), ..., (v).

8.3. Show that if A is a torsion group then $A \cong \operatorname{Tor}(A, \mathbb{Q}/\mathbb{Z})$; and that, in general, $\operatorname{Tor}(A, \mathbb{Q}/\mathbb{Z})$ embeds naturally as a subgroup of A. Identify this subgroup.

8.4. Show that if A and B are abelian groups and if $T(A)$, $T(B)$ are their torsion subgroups, then
$$\operatorname{Tor}(A, B) \cong \operatorname{Tor}(T(A), T(B)).$$
Show that $m \operatorname{Tor}(A, B) = 0$ if $m\, T(A) = 0$.

8.5. Show that Tor is additive in each variable.

8.6. Show that Tor respects direct limits over directed sets.

8.7. Show that the abelian group A is flat if and only if it is torsion-free.

8.8. Show that A' is pure in A if and only if $A' \otimes G \to A \otimes G$ is a monomorphism for all G (see Exercise I. 1.7).

8.9. Show that $\operatorname{Tor}^{\Lambda}(A, B)$ can be computed using a flat presentation of A; that is, if $R \overset{\mu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} A$ with P flat, then
$$\operatorname{Tor}^{\Lambda}(A, B) \cong \ker(\mu_* : R\otimes_{\Lambda} B \to P\otimes_{\Lambda} B).$$