5. Two Exact Sequences

Here we shall deduce two exact sequences connecting Hom and Ext. We start with the following very useful lemma.

Lemma 5.1. *Let the following commutative diagram have exact rows.*

$$\begin{array}{ccccccc} & & A & \xrightarrow{\mu} & B & \xrightarrow{\varepsilon} & C & \longrightarrow 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & \\ 0 & \longrightarrow & A' & \xrightarrow{\mu'} & B' & \xrightarrow{\varepsilon'} & C' & \end{array}$$

Then there is a "connecting homomorphism" $\omega : \ker\gamma \to \operatorname{coker}\alpha$ *such that the following sequence is exact:*

$$\ker\alpha \xrightarrow{\mu_*} \ker\beta \xrightarrow{\varepsilon_*} \ker\gamma \xrightarrow{\omega} \operatorname{coker}\alpha \xrightarrow{\mu'_*} \operatorname{coker}\beta \xrightarrow{\varepsilon'_*} \operatorname{coker}\gamma\,. \quad (5.1)$$

If μ *is monomorphic, so is* μ_* *: if* ε' *is epimorphic, so is* ε'_*.

Proof. It is very easy to see – and we leave the verification to the reader – that the final sentence holds and that we have exact sequences

$$\ker\alpha \xrightarrow{\mu_*} \ker\beta \xrightarrow{\varepsilon_*} \ker\gamma\,,$$

$$\operatorname{coker}\alpha \xrightarrow{\mu'_*} \operatorname{coker}\beta \xrightarrow{\varepsilon'_*} \operatorname{coker}\gamma\,.$$

It therefore remains to show that there exists a homomorphism $\omega : \ker\gamma \to \operatorname{coker}\alpha$ "connecting" these two sequences. In fact, ω is defined as follows.

Let $c \in \ker\gamma$, choose $b \in B$ with $\varepsilon b = c$. Since $\varepsilon'\beta b = \gamma\varepsilon b = \gamma c = 0$ there exists $a' \in A'$ with $\beta b = \mu' a'$. Define $\omega(c) = [a']$, the coset of a' in $\operatorname{coker}\alpha$.

We show that ω is well defined, that is, that $\omega(c)$ is independent of the choice of b. Indeed, let $\bar{b} \in B$ with $\varepsilon\bar{b} = c$, then $\bar{b} = b + \mu a$ and

$$\beta(b + \mu a) = \beta b + \mu'\alpha a\,.$$

Hence $\bar{a}' = a' + \alpha a$, thus $[\bar{a}'] = [a']$. Clearly ω is a homomorphism.

Next we show exactness at $\ker\gamma$. If $c \in \ker\gamma$ is of the form εb for $b \in \ker\beta$, then $0 = \beta b = \mu' a'$, hence $a' = 0$ and $\omega(c) = 0$. Conversely, let $c \in \ker\gamma$ with $\omega(c) = 0$. Then $c = \varepsilon b$, $\beta b = \mu' a'$ and there exists $a \in A$ with $\alpha a = a'$. Consider $\bar{b} = b - \mu a$. Clearly $\varepsilon\bar{b} = c$, but

$$\beta\bar{b} = \beta b - \beta\mu a = \beta b - \mu' a' = 0\,,$$

hence $c \in \ker\gamma$ is of the form $\varepsilon\bar{b}$ with $\bar{b} \in \ker\beta$.

Finally we prove exactness at $\operatorname{coker}\alpha'$. Let $\omega(c) = [a'] \in \operatorname{coker}\alpha$. Thus $c = \varepsilon b$, $\beta b = \mu' a'$, and $\mu'_*[a'] = [\mu' a'] = [\beta b] = 0$. Conversely, let $[a'] \in \operatorname{coker}\alpha$ with $\mu'_*[a'] = 0$. Then $\mu' a' = \beta b$ for some $b \in B$ and $c = \varepsilon b \in \ker\gamma$. Thus $[a'] = \omega(c)$. □