1.3. Prove the duals of Lemmas 1.1, 1.2, 1.3.
1.4. Show that the class of the split extension in $E(A, B)$ is preserved under the induced maps.
1.5. Prove: If P is projective, $E(P, B)$ contains only one element.
1.6. Prove: If I is injective, $E(A, I)$ contains only one element.
1.7. Show that $E(A, B_1 \oplus B_2) \cong E(A, B_1) \times E(A, B_2)$. Is there a corresponding formula with respect to the first variable?
1.8. Prove Theorem 1.4 using explicit constructions of pull-back and push-out.

2. The Functor Ext

In the previous section we have defined a bifunctor $E(-,-)$ from the category of Λ-modules to the categories of sets. In this section we shall define another bifunctor $\mathrm{Ext}_\Lambda(-,-)$ to the category of abelian groups, and eventually compare the two.

A short exact sequence $R \overset{\mu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} A$ of Λ-modules with P projective is called a *projective presentation* of A. By Theorem I.2.2 such a presentation induces for a Λ-module B an exact sequence

$$\mathrm{Hom}_\Lambda(A, B) \overset{\varepsilon^*}{\rightarrowtail} \mathrm{Hom}_\Lambda(P, B) \overset{\mu^*}{\longrightarrow} \mathrm{Hom}_\Lambda(R, B)\,. \tag{2.1}$$

To the modules A and B, and to the chosen projective presentation of A we therefore can associate the abelian group

$$\mathrm{Ext}^\varepsilon_\Lambda(A, B) = \mathrm{coker}(\mu^* : \mathrm{Hom}_\Lambda(P, B) \to \mathrm{Hom}_\Lambda(R, B))\,.$$

The superscript ε is to remind the reader that the group is defined via a particular projective presentation of A. An element in $\mathrm{Ext}^\varepsilon_\Lambda(A, B)$ may be represented by a homomorphism $\varphi : R \to B$. The element represented by $\varphi : R \to B$ will be denoted by $[\varphi] \in \mathrm{Ext}^\varepsilon_\Lambda(A, B)$. Then $[\varphi_1] = [\varphi_2]$ if and only if $\varphi_1 - \varphi_2$ extends to P.

Clearly a homomorphism $\beta : B \to B'$ will map the sequence (2.1) into the corresponding sequence for B'. We thus get an induced map $\beta_* : \mathrm{Ext}^\varepsilon_\Lambda(A, B) \to \mathrm{Ext}^\varepsilon_\Lambda(A, B')$, which is easily seen to make $\mathrm{Ext}^\varepsilon_\Lambda(A, -)$ into a functor.

Next we will show that for two different projective presentations of A we obtain the "same" functor. Let $R' \overset{\mu'}{\rightarrowtail} P' \overset{\varepsilon'}{\twoheadrightarrow} A'$ and $R \overset{\mu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} A$ be projective presentations of A', A respectively. Let $\alpha : A' \to A$ be a homomorphism. Since P' is projective, there is a homomorphism $\pi : P' \to P$, inducing $\sigma : R' \to R$ such that the following diagram is commutative:

$$\begin{array}{ccccc} R' & \overset{\mu'}{\rightarrowtail} & P' & \overset{\varepsilon'}{\twoheadrightarrow} & A' \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\alpha} \\ R & \overset{\mu}{\rightarrowtail} & P & \overset{\varepsilon}{\twoheadrightarrow} & A \end{array}$$

We sometimes say that π *lifts* α.

Clearly π, together with σ, will induce a map

$$\pi^* : \mathrm{Ext}^{\varepsilon}_{\Lambda}(A, B) \to \mathrm{Ext}^{\varepsilon'}_{\Lambda}(A', B)$$

which plainly is natural in B. Thus every π gives rise to a natural transformation from $\mathrm{Ext}^{\varepsilon}_{\Lambda}(A, -)$ into $\mathrm{Ext}^{\varepsilon'}_{\Lambda}(A', -)$. In the following lemma we prove that this natural transformation depends only on $\alpha : A' \to A$ and not on the chosen $\pi : P' \to P$ lifting α.

Lemma 2.1. *π^* does not depend on the chosen $\pi : P' \to P$ but only on $\alpha : A' \to A$.*

Proof. Let $\pi_i : P' \to P$, $i = 1, 2$, be two homomorphisms lifting α and inducing $\sigma_i : R' \to R$, so that the following diagram is commutative for $i = 1, 2$

$$\begin{array}{ccccc} R' & \overset{\mu'}{\rightarrowtail} & P' & \overset{\varepsilon'}{\twoheadrightarrow} & A' \\ \downarrow{\scriptstyle \sigma_i} & & \downarrow{\scriptstyle \pi_i} & & \downarrow{\scriptstyle \alpha} \\ R & \overset{\mu}{\rightarrowtail} & P & \overset{\varepsilon}{\twoheadrightarrow} & A \end{array}$$

Consider $\pi_1 - \pi_2$; since π_1, π_2 induce the same map $\alpha : A' \to A$, $\pi_1 - \pi_2$ factors through a map $\tau : P' \to R$, such that $\pi_1 - \pi_2 = \mu\tau$. It follows that $\sigma_1 - \sigma_2 = \tau\mu'$. Thus, if $\varphi : R \to B$ is a representative of the element $[\varphi] \in \mathrm{Ext}^{\varepsilon}_{\Lambda}(A, B)$, we have $\pi_1^*[\varphi] = [\varphi\sigma_1] = [\varphi\sigma_2 + \varphi\tau\mu'] = [\varphi\sigma_2] = \pi_2^*[\varphi]$. □

To stress the independence from the choice of π we shall call the natural transformation $(\alpha; P', P) : \mathrm{Ext}^{\varepsilon}_{\Lambda}(A, -) \to \mathrm{Ext}^{\varepsilon'}_{\Lambda}(A', -)$, instead of π^*. Let $\alpha' : A'' \to A'$ and $\alpha : A' \to A$ be two homomorphisms and $R'' \rightarrowtail P'' \twoheadrightarrow A''$, $R' \rightarrowtail P' \twoheadrightarrow A'$, $R \rightarrowtail P \to A$ projective presentations of A'', A', A respectively. Let $\pi' : P'' \to P'$ lift α' and $\pi : P' \to P$ lift α. Then $\pi \circ \pi' : P'' \to P$ lifts $\alpha \circ \alpha'$; whence it follows that

$$(\alpha'; P'', P') \circ (\alpha; P', P) = (\alpha \circ \alpha'; P'', P). \tag{2.2}$$

Also, we have

$$(1_A; P, P) = 1. \tag{2.3}$$

This yields a proof of

Corollary 2.2. *Let $R \rightarrowtail P \overset{\varepsilon}{\twoheadrightarrow} A$ and $R' \rightarrowtail P' \overset{\varepsilon'}{\twoheadrightarrow} A$ be two projective presentations of A. Then*

$$(1_A; P', P) : \mathrm{Ext}^{\varepsilon}_{\Lambda}(A, -) \to \mathrm{Ext}^{\varepsilon'}_{\Lambda}(A, -)$$

is a natural equivalence.

Proof. Let $\pi : P \to P'$ and $\pi' : P' \to P$ both lift $1_A : A \to A$. By formulas (2.2) and (2.3) we obtain $(1_A; P, P') \circ (1_A; P', P) = (1_A; P, P) = 1 : \mathrm{Ext}^{\varepsilon}_{\Lambda}(A, -) \to \mathrm{Ext}^{\varepsilon}_{\Lambda}(A, -)$. Analogously, $(1_A; P', P) \circ (1_A; P, P') = 1$, whence the assertion. □

By this natural equivalence we are allowed to drop the superscript ε and to write, simply, $\mathrm{Ext}_\Lambda(A, B)$.

Of course, we want to make $\mathrm{Ext}_\Lambda(-, B)$ into a functor. It is obvious by now that given $\alpha : A' \to A$ we can define an induced map α^* as follows: Choose projective presentations $R' \rightarrowtail P' \overset{\varepsilon'}{\twoheadrightarrow} A'$ and $R \rightarrowtail P \overset{\varepsilon}{\twoheadrightarrow} A$ of A', A respectively, and let $\alpha^* = (\alpha; P', P) : \mathrm{Ext}^\varepsilon_\Lambda(A, B) \to \mathrm{Ext}^{\varepsilon'}_\Lambda(A', B)$. Formulas (2.2), (2.3) establish the facts that this definition is compatible with the natural equivalences of Corollary 2.2 and that $\mathrm{Ext}_\Lambda(-, B)$ becomes a (contravariant) functor. We leave it to the reader to prove the *bi*-functoriality part in the following theorem.

Theorem 2.3. $\mathrm{Ext}_\Lambda(-, -)$ *is a bifunctor from the category of Λ-modules to the category of abelian groups. It is contravariant in the first, and covariant in the second variable.* □

Instead of regarding $\mathrm{Ext}_\Lambda(A, B)$ as an abelian group, we clearly can regard it just as a set. We thus obtain a *set*-valued bifunctor which – for convenience – we shall still call $\mathrm{Ext}_\Lambda(-, -)$.

Theorem 2.4. *There is a natural equivalence of set-valued bifunctors* $\eta : E(A, B) \xrightarrow{\sim} \mathrm{Ext}_\Lambda(A, B)$.

Proof. We first define an isomorphism of sets

$$\eta : E(A, B) \xrightarrow{\sim} \mathrm{Ext}^\varepsilon_\Lambda(A, B),$$

natural in B, where $R \overset{\mu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} A$ is a fixed projective presentation of A. We will then show that η is natural in A.

Given an element in $E(A, B)$, represented by the extension $B \overset{\kappa}{\rightarrowtail} E \overset{\nu}{\twoheadrightarrow} A$, we form the diagram

$$\begin{array}{ccccc} R & \overset{\mu}{\rightarrowtail} & P & \overset{\varepsilon}{\twoheadrightarrow} & A \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\varphi} & & \| \\ B & \overset{\kappa}{\rightarrowtail} & E & \overset{\nu}{\twoheadrightarrow} & A \end{array}$$

The homomorphism $\psi : R \to B$ defines an element $[\psi] \in \mathrm{Ext}^\varepsilon_\Lambda(A, B) = \mathrm{coker}(\mu^* : \mathrm{Hom}_\Lambda(P, B) \to \mathrm{Hom}_\Lambda(R, B))$. We claim that this element does not depend on the particular $\varphi : P \to E$ chosen. Thus let $\varphi_i : P \to E$, $i = 1, 2$, be two maps inducing $\psi_i : R \to B$, $i = 1, 2$. Then $\varphi_1 - \varphi_2$ factors through $\tau : P \to B$, i.e., $\varphi_1 - \varphi_2 = \kappa\tau$. It follows that $\psi_1 - \psi_2 = \tau\mu$, whence $[\psi_1] = [\psi_2 + \tau\mu] = [\psi_2]$.

Since two representatives of the same element in $E(A, B)$ obviously induce the same element in $\mathrm{Ext}^\varepsilon_\Lambda(A, B)$, we have defined a map $\eta : E(A, B) \to \mathrm{Ext}^\varepsilon_\Lambda(A, B)$. We leave it to the reader to prove the naturality of η with respect to B.

Conversely, given an element in $\mathrm{Ext}^\varepsilon_\Lambda(A, B)$, we represent this element by a homomorphism $\psi : R \to B$. Taking the push-out of (ψ, μ) we obtain the diagram

$$\begin{array}{ccccc} R & \xrightarrow{\mu} & P & \xrightarrow{\varepsilon} & A \\ \downarrow{\scriptstyle\psi} & & \vdots{\scriptstyle\varphi} & & \| \\ B & \xrightarrow{\kappa} & E & \xrightarrow{\nu} & A \end{array}$$

By the dual of Lemma 1.2 the bottom row $B \xrightarrow{\kappa} E \xrightarrow{\nu} A$ is an extension. We claim that the equivalence class of this extension is independent of the particular representative $\psi: R \to B$ chosen. Indeed another representative $\psi': R \to B$ has the form $\psi' = \psi + \tau\mu$ where $\tau: P \to B$. The reader may check that the diagram

$$\begin{array}{ccccc} R & \xrightarrow{\mu} & P & \xrightarrow{\varepsilon} & A \\ \downarrow{\scriptstyle\psi'} & & \downarrow{\scriptstyle\varphi'} & & \| \\ B & \xrightarrow{\kappa} & E & \xrightarrow{\nu} & A \end{array}$$

with $\varphi' = \varphi + \kappa\tau$ is commutative. By the dual of Lemma 1.3 the left hand square is a push-out diagram, whence it follows that the extension we arrive at does not depend on the representative. We thus have defined a map

$$\xi: \mathrm{Ext}^{\varepsilon}_{\Lambda}(A, B) \to E(A, B)$$

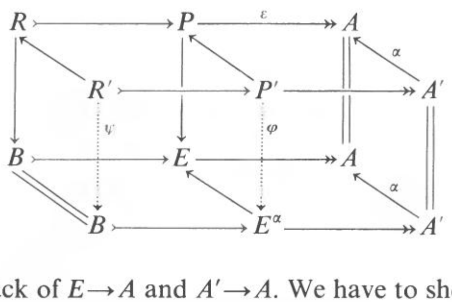
which is easily seen to be natural in B.

Using Lemma 1.3 it is easily proved that η, ξ are inverse to each other. We thus have an equivalence

$$\eta: E(A, B) \xrightarrow{\sim} \mathrm{Ext}^{\varepsilon}_{\Lambda}(A, B)$$

which is natural in B.

Note that η might conceivably depend upon the projective presentation of A. However we show that this cannot be the case by the following (3-dimensional) diagram, which shows also the naturality of η in A.



E^{α} is the pull-back of $E \to A$ and $A' \to A$. We have to show the existence of homomorphisms $\varphi: P' \to E^{\alpha}$, $\psi: R' \to B$ such that all faces are commutative. Since the maps $P' \to E \to A$ and $P' \to A' \to A$ agree they define a homomorphism $\varphi: P' \to E^{\alpha}$, into the pull-back. Then φ induces

$\psi: R' \to B$, and trivially all faces are commutative. (To see that $R' \to R \to B$ coincides with ψ, compose each with $B \rightarrowtail E$.) We therefore arrive at a commutative diagram

$$\begin{array}{ccc} E(A,B) & \xrightarrow{\alpha^*} & E(A',B) \\ \eta\downarrow\uparrow\xi & & \eta\downarrow\uparrow\xi \\ \operatorname{Ext}_\Lambda^\varepsilon(A,B) & \xrightarrow{\alpha^*} & \operatorname{Ext}_\Lambda^{\varepsilon'}(A',B) \end{array}$$

For $A' = A$, $\alpha = 1_A$ this shows that η is independent of the chosen projective presentation. In general it shows that η and ξ are natural in A. □

Corollary 2.5. *The set $E(A, B)$ of equivalence classes of extensions has a natural abelian group structure.*

Proof. This is obvious, since $\operatorname{Ext}_\Lambda(A, B)$ carries a natural abelian group structure and since $\eta: E(-,-) \xrightarrow{\sim} \operatorname{Ext}_\Lambda(-,-)$ is a natural equivalence. □

We leave as exercises (see Exercises 2.5 to 2.7) the direct description of the group structure in $E(A, B)$. However we shall exhibit here the neutral element of this group. Consider the diagram

$$\begin{array}{ccccc} R & \xrightarrow{\mu} & P & \xrightarrow{\varepsilon} & A \\ \downarrow\psi & & \downarrow\varphi & & \| \\ B & \xrightarrow{\kappa} & E & \xrightarrow{\nu} & A \end{array}$$

The extension $B \rightarrowtail E \twoheadrightarrow A$ represents the neutral element in $E(A, B)$ if and only if $\psi: R \to B$ is the restriction of a homomorphism $\tau: P \to B$, i.e., if $\psi = \tau\mu$. The map $(\varphi - \kappa\tau)\mu: R \to E$ therefore is the zero map, so that $\varphi - \kappa\tau$ factors through A, defining a map $\sigma: A \to E$ with $\varphi - \kappa\tau = \sigma\varepsilon$. Since $\nu(\varphi - \kappa\tau) = \varepsilon$, σ is a right inverse to ν. Thus the extension $B \rightarrowtail E \twoheadrightarrow A$ splits. Conversely if $B \overset{\kappa}{\rightarrowtail} E \overset{\nu}{\twoheadrightarrow} A$ splits, the left inverse of κ is a map $E \to B$ which if composed with $\varphi: P \to E$ yields τ.

We finally note

Proposition 2.6. *If P is projective and I injective, then* $\operatorname{Ext}_\Lambda(P, B) = 0 = \operatorname{Ext}_\Lambda(A, I)$ *for all Λ-modules A, B.*

Proof. By Theorem 2.4 $\operatorname{Ext}_\Lambda(P, B)$ is in one-to-one correspondence with the set $E(P, B)$, consisting of classes of extensions of the form $B \rightarrowtail E \twoheadrightarrow P$. By Lemma I.4.5 short exact sequences of this form split. Hence $E(P, B)$ contains only one element, the zero element. For the other assertion one proceeds dually. □

Of course, we could prove this proposition directly, without involving Theorem 2.4.

Exercises:

2.1. Prove that $\operatorname{Ext}_\Lambda(-,-)$ is a bifunctor.

2.2. Suppose A is a right Γ-left Λ-bimodule. Show that $\operatorname{Ext}_\Lambda(A, B)$ has a left-Γ-module structure which is natural in B.

2.3. Suppose B is a right Γ-left Λ-bimodule. Show that $\operatorname{Ext}_\Lambda(A, B)$ has a right Γ-module structure, which is natural in A.

2.4. Suppose Λ commutative. Show that $\operatorname{Ext}_\Lambda(A, B)$ has a natural (in A and B) Λ-module structure.

2.5. Show that one can define an addition in $E(A, B)$ as follows: Let $B \rightarrowtail E_1 \twoheadrightarrow A$, $B \rightarrowtail E_2 \twoheadrightarrow A$ be representatives of two elements ξ_1, ξ_2 in $E(A, B)$. Let $\Delta_B : B \to B \oplus B$ be the map defined by $\Delta_B(b) = (b, b)$, $b \in B$, and let $\nabla_A : A \oplus A \to A$ be the map defined by $\nabla_A(a_1, a_2) = a_1 + a_2$, $a_1, a_2 \in A$. Define the sum $\xi_1 + \xi_2$ by

$$\xi_1 + \xi_2 = E(\Delta_B, \nabla_A)\,(B \oplus B \rightarrowtail E_1 \oplus E_2 \twoheadrightarrow A \oplus A)\,.$$

2.6. Show that if $\alpha_1, \alpha_2 : A' \to A$, then

$$(\alpha_1 + \alpha_2)^* = \alpha_1^* + \alpha_2^* : E(A, B) \to E(A', B)\,,$$

using the addition given in Exercise 2.5. Deduce that $E(A, B)$ admits additive inverses (without using Theorem 2.4).

2.7. Show that the addition defined in Exercise 2.5 is commutative and associative (without using Theorem 2.4). [Thus $E(A, B)$ is an abelian group.]

2.8. Let $\mathbb{Z}_4 \rightarrowtail \mathbb{Z}_{16} \twoheadrightarrow \mathbb{Z}_4$ be the evident exact sequence. Construct its inverse in $E(\mathbb{Z}_4, \mathbb{Z}_4)$.

2.9. Show the group table of $E(\mathbb{Z}_8, \mathbb{Z}_{12})$.

3. Ext Using Injectives

Given two Λ-modules A, B, we defined in Section 2 a group $\operatorname{Ext}_\Lambda(A, B)$ by using a projective presentation $R \overset{\mu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} A$ of A:

$$\operatorname{Ext}_\Lambda(A, B) = \operatorname{coker}(\mu^* : \operatorname{Hom}_\Lambda(P, B) \to \operatorname{Hom}_\Lambda(R, B))\,.$$

Here we consider the dual procedure: Choose an *injective presentation* of B, i.e. an exact sequence $B \overset{\nu}{\rightarrowtail} I \overset{\eta}{\twoheadrightarrow} S$ with I injective, and define the group $\overline{\operatorname{Ext}}^\nu_\Lambda(A, B)$ as the cokernel of the map $\eta_* : \operatorname{Hom}_\Lambda(A, I) \to \operatorname{Hom}_\Lambda(A, S)$. Dualizing the proofs of Lemma 2.1, Corollary 2.2, and Theorem 2.3 one could show that $\overline{\operatorname{Ext}}^\nu_\Lambda(A, B)$ does not depend upon the chosen injective presentation, and that $\overline{\operatorname{Ext}}_\Lambda(-,-)$ can be made into a bifunctor, covariant in the second, contravariant in the first variable. Also, by dualizing the proof of Theorem 2.4 one proves that there is a natural equivalence of set-valued bifunctors between $E(-,-)$ and $\overline{\operatorname{Ext}}_\Lambda(-,-)$.

Here we want to give a different proof of the facts mentioned above which has the advantage of yielding yet another description of $E(-,-)$. In contrast to $\operatorname{Ext}_\Lambda(-,-)$ and $\overline{\operatorname{Ext}}_\Lambda(-,-)$, the new description will be symmetric in A and B. Also, this proof establishes immediately that $\mathrm{Ext}_\Lambda(A,B)$ and $\overline{\mathrm{Ext}}_\Lambda(A,B)$ are isomorphic as abelian groups. First let us state the following lemma, due to J. Lambek (see [32]).

Lemma 3.1. *Let*

$$\begin{array}{ccccc} A' & \xrightarrow{\alpha_1} & A & \xrightarrow{\alpha_2} & A'' \\ \downarrow{\scriptstyle\psi} & \Sigma_1 & \downarrow{\scriptstyle\varphi} & \Sigma_2 & \downarrow{\scriptstyle\theta} \\ B' & \xrightarrow{\beta_1} & B & \xrightarrow{\beta_2} & B'' \end{array} \tag{3.1}$$

be a commutative diagram with exact rows. Then φ induces an isomorphism

$$\Phi : \ker\theta\alpha_2/(\ker\alpha_2 + \ker\varphi) \xrightarrow{\sim} (\operatorname{im}\varphi \cap \operatorname{im}\beta_1)/\operatorname{im}\varphi\alpha_1 .$$

Proof. First we show that φ induces a homomorphism of this kind. Let $x \in \ker\theta\alpha_2$; plainly $\varphi x \in \operatorname{im}\varphi$. Since $0 = \theta\alpha_2 x = \beta_2\varphi x$, $\varphi x \in \operatorname{im}\beta_1$. If $x \in \ker\alpha_2$, then $x \in \operatorname{im}\alpha_1$, and $\varphi x \in \operatorname{im}\varphi\alpha_1$. If $x \in \ker\varphi$, $\varphi x = 0$. Thus Φ is well-defined. Clearly Φ is a homomorphism. To show it is epimorphic, let $y \in \operatorname{im}\varphi \cap \operatorname{im}\beta_1$. There exists $x \in A$ with $\varphi x = y$. Since

$$\theta\alpha_2 x = \beta_2\varphi x = \beta_2 y = 0 ,$$

$x \in \ker\theta\alpha_2$. Finally we show that Φ is monomorphic. Suppose $x \in \ker\theta\alpha_2$, such that $\varphi x \in \operatorname{im}\varphi\alpha_1$, i.e. $\varphi x = \varphi\alpha_1 z$ for some $z \in A'$. Then $x = \alpha_1 z + t$, where $t \in \ker\varphi$. It follows that $x \in \ker\alpha_2 + \ker\varphi$. □

To facilitate the notation we introduce some terminology.

Definition. Let Σ be a commutative square of Λ-modules

$$\begin{array}{ccc} A' & \xrightarrow{\alpha} & A \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\varphi} \\ B' & \xrightarrow{\beta} & B \end{array}$$

We then write

$$\operatorname{Im}\Sigma = \operatorname{im}\varphi \cap \operatorname{im}\beta/\operatorname{im}\varphi\alpha ,$$

$$\operatorname{Ker}\Sigma = \ker\varphi\alpha/\ker\alpha + \ker\psi .$$

With this notation Lemma 3.1 may be stated in the following form:

If the diagram (3.1) *has exact rows, then φ induces an isomorphism* $\Phi : \operatorname{Ker}\Sigma_2 \xrightarrow{\sim} \operatorname{Im}\Sigma_1$.

Proposition 3.2. *For any projective presentation $R \overset{\mu}{\rightarrowtail} P \overset{\varepsilon}{\twoheadrightarrow} A$ of A and any injective presentation $B \overset{\nu}{\rightarrowtail} I \overset{\eta}{\twoheadrightarrow} S$ of B, there is an isomorphism*

$$\sigma : \mathrm{Ext}^{\varepsilon}_\Lambda(A,B) \xrightarrow{\sim} \mathrm{Ext}^{\nu}_\Lambda(A,B) .$$

Proof. Consider the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccc}
\operatorname{Hom}_\Lambda(A,B) & \rightarrowtail & \operatorname{Hom}_\Lambda(A,I) & \to & \operatorname{Hom}_\Lambda(A,S) & \twoheadrightarrow & \overline{\operatorname{Ext}}{}^{\nu}_\Lambda(A,B) \\
\downarrow & & \downarrow & \Sigma_2 & \downarrow & \Sigma_1 & \downarrow \\
\operatorname{Hom}_\Lambda(P,B) & \rightarrowtail & \operatorname{Hom}_\Lambda(P,I) & \to & \operatorname{Hom}_\Lambda(P,S) & \longrightarrow & 0 \\
\downarrow & \Sigma_4 & \downarrow & \Sigma_3 & \downarrow & & \\
\operatorname{Hom}_\Lambda(R,B) & \rightarrowtail & \operatorname{Hom}_\Lambda(R,I) & \to & \operatorname{Hom}_\Lambda(R,S) & & \\
\downarrow & \Sigma_5 & \downarrow & & & & \\
\operatorname{Ext}^{\varepsilon}_\Lambda(A,B) & \longrightarrow & 0 & & & &
\end{array} \tag{3.2}$$

The reader easily checks that $\operatorname{Ker}\Sigma_1 = \overline{\operatorname{Ext}}{}^{\nu}_\Lambda(A,B)$ and $\operatorname{Ker}\Sigma_5 = \operatorname{Ext}^{\varepsilon}_\Lambda(A,B)$. Applying Lemma 3.1 repeatedly we obtain

$$\overline{\operatorname{Ext}}{}^{\nu}_\Lambda(A,B) = \operatorname{Ker}\Sigma_1 \cong \operatorname{Im}\Sigma_2 \cong \operatorname{Ker}\Sigma_3 \cong \operatorname{Im}\Sigma_4 \cong \operatorname{Ker}\Sigma_5 = \operatorname{Ext}^{\varepsilon}_\Lambda(A,B)\,. \quad \square$$

Thus for *any* injective presentation of B, $\overline{\operatorname{Ext}}{}^{\nu}_\Lambda(A,B)$ is isomorphic to $\operatorname{Ext}^{\varepsilon}_\Lambda(A,B)$. We thus are allowed to drop the superscript ν and to write $\overline{\operatorname{Ext}}_\Lambda(A,B)$. Let $\beta: B \to B'$ be a homomorphism and let $B' \overset{\nu'}{\rightarrowtail} I' \twoheadrightarrow S'$ be an injective presentation. It is easily seen that if $\tau: I \to I'$ is a map inducing β the diagram (3.2) is mapped into the corresponding diagram for $B' \overset{\nu'}{\rightarrowtail} I' \twoheadrightarrow S'$. Therefore we obtain an induced homomorphism

$$\beta_*: \overline{\operatorname{Ext}}_\Lambda(A,B) \to \overline{\operatorname{Ext}}_\Lambda(A,B')$$

which agrees via the isomorphism defined above with the induced homomorphism $\beta_*: \operatorname{Ext}_\Lambda(A,B) \to \operatorname{Ext}_\Lambda(A,B')$.

Analogously one defines an induced homomorphism in the first variable. With these definitions of induced maps $\overline{\operatorname{Ext}}_\Lambda(-,-)$ becomes a bifunctor, and σ becomes a natural equivalence. We thus have

Corollary 3.3. $\overline{\operatorname{Ext}}_\Lambda(-,-)$ *is a bifunctor, contravariant in the first, covariant in the second variable. It is naturally equivalent to* $\operatorname{Ext}_\Lambda(-,-)$ *and therefore to* $E(-,-)$. $\square$

We sometimes express the natural equivalence between $\operatorname{Ext}_\Lambda(-,-)$ and $\overline{\operatorname{Ext}}_\Lambda(-,-)$ by saying that Ext is *balanced*.

Finally the above proof also yields a symmetric description of Ext from (3.2), namely:

Corollary 3.4. $\operatorname{Ext}_\Lambda(A,B) \cong \operatorname{Ker}\Sigma_3$. $\square$

In view of the above results we shall use only one notation, namely $\operatorname{Ext}_\Lambda(-,-)$ for the equivalent functors $E(-,-)$, $\operatorname{Ext}_\Lambda(-,-)$, $\overline{\operatorname{Ext}}_\Lambda(-,-)$.

Exercises:

3.1. Show that, if Λ is a principal ideal domain (p.i.d.), then an epimorphism $\beta: B \twoheadrightarrow B'$ induces an epimorphism $\beta_*: \operatorname{Ext}_\Lambda(A, B) \to \operatorname{Ext}_\Lambda(A, B')$. State and prove the dual.

3.2. Prove that $\operatorname{Ext}_{\mathbb{Z}}(A, \mathbb{Z}) \neq 0$ if A has elements of finite order.

3.3. Compute $\operatorname{Ext}_{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z})$, using an *injective* presentation of $\mathbb{Z}$.

3.4. Show that $\operatorname{Ext}_{\mathbb{Z}}(A, \operatorname{Ext}_{\mathbb{Z}}(B, C)) \cong \operatorname{Ext}_{\mathbb{Z}}(B, \operatorname{Ext}_{\mathbb{Z}}(A, C))$ when A, B, C are finitely-generated abelian groups.

3.5. Let the natural equivalences $\eta: E(-,-) \to \operatorname{Ext}_\Lambda(-,-)$ be defined by Theorem 2.4, $\sigma: \overline{\operatorname{Ext}}_\Lambda(-,-) \to \operatorname{Ext}_\Lambda(-,-)$ by Proposition 3.2, and

$$\bar{\eta}: E(-,-) \to \overline{\operatorname{Ext}}_\Lambda(-,-)$$

by dualizing the proof of Theorem 2.4. Show that $\sigma \circ \eta = \bar{\eta}$.

4. Computation of some Ext-Groups

We start with the following

Lemma 4.1. (i) $\operatorname{Ext}_\Lambda\left(\bigoplus_i A_i, B\right) \cong \prod_i \operatorname{Ext}_\Lambda(A_i, B)$,

(ii) $\operatorname{Ext}_\Lambda\left(A, \prod_j B_j\right) \cong \prod_j \operatorname{Ext}_\Lambda(A, B_j)$.

Proof. We only prove assertion (i), leaving the other to the reader. For each i in the index set we choose a projective presentation $R_i \rightarrowtail P_i \twoheadrightarrow A_i$ of A_i. Then $\bigoplus_i R_i \rightarrowtail \bigoplus_i P_i \twoheadrightarrow \bigoplus_i A_i$ is a projective presentation of $\bigoplus_i A_i$. Using Proposition I.3.4 we obtain the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
\operatorname{Hom}_\Lambda\left(\bigoplus_i A_i, B\right) & \rightarrowtail & \operatorname{Hom}_\Lambda\left(\bigoplus_i P_i, B\right) & \to & \operatorname{Hom}_\Lambda\left(\bigoplus_i R_i, B\right) & \twoheadrightarrow & \operatorname{Ext}_\Lambda\left(\bigoplus_i A_i, B\right) \\
\downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\
\prod_i \operatorname{Hom}_\Lambda(A_i, B) & \rightarrowtail & \prod_i \operatorname{Hom}_\Lambda(P_i, B) & \to & \prod_i \operatorname{Hom}_\Lambda(R_i, B) & \twoheadrightarrow & \prod_i \operatorname{Ext}_\Lambda(A_i, B)
\end{array}$$

whence the result. □

The reader may prefer to prove assertion (i) by using an injective presentation of B. Indeed in doing so it becomes clear that the two assertions of Lemma 4.1 are dual to each other.

In the remainder of this section we shall compute $\operatorname{Ext}_{\mathbb{Z}}(A, B)$ for A, B finitely-generated abelian groups. In view of Lemma 4.1 it is enough to consider the case where A, B are cyclic.

To facilitate the notation we shall write $\operatorname{Ext}(A, B)$ (for $\operatorname{Ext}_{\mathbb{Z}}(A, B)$) and $\operatorname{Hom}(A, B)$ (for $\operatorname{Hom}_{\mathbb{Z}}(A, B)$), whenever the groundring is the ring of integers.

Since $\mathbb{Z}$ is projective, one has

$$\operatorname{Ext}(\mathbb{Z}, \mathbb{Z}) = 0 = \operatorname{Ext}(\mathbb{Z}, \mathbb{Z}_q)$$

by Proposition 2.6. To compute $\operatorname{Ext}(\mathbb{Z}_r, \mathbb{Z})$ and $\operatorname{Ext}(\mathbb{Z}_r, \mathbb{Z}_q)$ we use the projective presentation

$$\mathbb{Z} \overset{\mu}{\rightarrowtail} \mathbb{Z} \overset{\varepsilon}{\twoheadrightarrow} \mathbb{Z}_r$$

where μ is multiplication by r. We obtain the exact sequence

$$\begin{array}{ccccccc} \operatorname{Hom}(\mathbb{Z}_r, \mathbb{Z}) & \rightarrowtail & \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}) & \xrightarrow{\mu^*} & \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}) & \twoheadrightarrow & \operatorname{Ext}(\mathbb{Z}_r, \mathbb{Z}) \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\ 0 & \rightarrowtail & \mathbb{Z} & \xrightarrow{\mu^*} & \mathbb{Z} & & \end{array}$$

Since μ^* is again multiplication by r we obtain

$$\operatorname{Ext}(\mathbb{Z}_r, \mathbb{Z}) \cong \mathbb{Z}_r .$$

Also the exact sequence

$$\begin{array}{ccccccc} \operatorname{Hom}(\mathbb{Z}_r, \mathbb{Z}_q) & \rightarrowtail & \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}_q) & \xrightarrow{\mu^*} & \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}_q) & \twoheadrightarrow & \operatorname{Ext}(\mathbb{Z}_r, \mathbb{Z}_q) \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\ \mathbb{Z}_{(r,q)} & \rightarrowtail & \mathbb{Z}_q & \xrightarrow{\mu^*} & \mathbb{Z}_q & & \end{array}$$

yields, since μ^* is multiplication by r,

$$\operatorname{Ext}(\mathbb{Z}_r, \mathbb{Z}_q) \cong \mathbb{Z}_{(r,q)}$$

where (r, q) denotes the greatest common divisor of r and q.

Exercises:

4.1. Show that there are p non-equivalent extensions $\mathbb{Z}_p \rightarrowtail E \twoheadrightarrow \mathbb{Z}_p$ for p a prime, but only two non-isomorphic groups E, namely $\mathbb{Z}_p \oplus \mathbb{Z}_p$ and $\mathbb{Z}_{p^2}$. How does this come about?

4.2. Classify the extension classes $[E]$, given by

$$\mathbb{Z}_m \rightarrowtail E \twoheadrightarrow \mathbb{Z}_n$$

under automorphisms of $\mathbb{Z}_m$ and $\mathbb{Z}_n$.

4.3. Show that if A is a finitely-generated abelian group such that $\operatorname{Ext}(A, \mathbb{Z}) = 0$, $\operatorname{Hom}(A, \mathbb{Z}) = 0$, then $A = 0$.

4.4. Show that $\operatorname{Ext}(A, \mathbb{Z}) \cong A$ if A is a finite abelian group.

4.5. Show that there is a natural equivalence of functors $\operatorname{Hom}(-, \mathbb{Q}/\mathbb{Z}) \cong \operatorname{Ext}(-, \mathbb{Z})$ if both functors are restricted to the category of torsion abelian groups.

4.6. Show that extensions of finite abelian groups of relatively prime order split.