Definition (Additive Category) : An additive category $\underline{A}$ is a category with zero object in which any two objects have a product and in which the morphism sets $\underline{A}(A,B)$ are abelian groups such that the composition

$$\underline{A}(A,B) \times \underline{A}(B,C) \to \underline{A}(A,C)$$

is <u>bilinear</u>.

Definition (Abelian Category) An additive category in which

(i) every morphism has a kernel and a cokernel

(ii) every monomorphism is the kernel of its cokernel; every epimorphism is the cokernel of its kernel.

(iii) every morphism is expressible as the composite of an epimorphism and a monomorphism.

Main Example : Module categories.

Basically, exact sequences should make sense.

Free and Projective Resolutions.

Every module is an epimorphic image of a free module (hence also projective). Just choose set of gen. $\{g_i\}$ for M and take a free mod. on $\{e_i\}$ $e_i \to g_i$..

This makes it easy to compare any module to free modules: If $\alpha : F_0 \twoheadrightarrow M$ is an epi, then we may ~~say~~ that F_0 differs from M by the module $\ker \alpha$.

We can express M in terms of free modules "better" by mapping a free module F_1 onto $\ker \alpha$. Taking φ_1 to be the composite:

$$F_1 \twoheadrightarrow \ker\alpha \rightarrowtail F_0$$

we say that instead $M = \operatorname{Coker} \varphi_1 : F_1 \to F_0$.

$\ker \varphi_1$ is still lurking around: take another free:

$$F_2 \twoheadrightarrow \ker \varphi_1 \hookrightarrow F_1$$

Can think of M as given by the sequence of free modules $F_2 \to F_1 \to F_0$.

$\ker \varphi_2$ still may not be free. Repeat (possibly indefinitely) to get free modules & a sequence:

$$F: \quad \cdots \to F_{i+1} \xrightarrow{\varphi_{i+1}} F_i \xrightarrow{\varphi_i} F_{i-1} \to \cdots \xrightarrow{\varphi_1} F_0$$

This is a "Free Resolution". If we can do with projectives it is called a "Projective Resolution".