Topological space for a scheme

Among the topological spaces $\mathrm{Spec}(\mathbb{Z})$,$\mathrm{Spec}(\mathbb{F}_2[x])$,$\mathrm{Spec}(\mathbb{Z}[x])$ two of them are homeomorphic and the third is different.

Identify the pair and explain your answer.

Solution

Since $\mathbb{Z}$ and $\mathbb{F}_2$ are *principal ideal domains*, we see that any ideal is generated by a single element. If the ideal is not $\{0\}$ then it is contained in finitely many prime ideals (which are also maximal).

However, for each prime integer p, the ideal $\langle p, x\rangle$ is maximal ideal containing $\langle x\rangle$. Hence, the topological space $\mathrm{Spec}(\mathbb{Z}[x])$ contains a *proper* closed subset $\mathrm{Spec}(\mathbb{Z}[x]/\langle x\rangle)$ which is infinite.

The topological spaces $\mathrm{Spec}(\mathbb{Z})$ and $\mathrm{Spec}(\mathbb{F}_2)$ are *countable*, contain a special point g corresponding to the prime ideal $\{0\}$. Every closed set that does not contain g is finite and the closed set that contains g is the whole space. Thus, these spaces are homeomorphic under *any* bijection that sends g to g.