Schemes and Patching

Write suitable Quasi-affine schemes U_i and $V_{i,j}$ and morphisms $V_{i,j} \to U_i$ and $V_{i,j} \to U_j$ that describe the quasi-projective scheme $X = P(x_0, x_1, x_2; \ ; x_0x_2, x_1)$.

The quasi-affine schemes should be given in the form $Q(x_1, \ldots, x_p; f_1, \ldots, f_q; g_1, \ldots, g_r)$ by specifying the variables and the polynomials f_i's and g_j's in those variables.

The morphisms should be given in the form $x_1 \mapsto h_1; \ldots; x_p \mapsto h_p$ by specifying the polynomials $h_1, \ldots, h_p$.

Solution

Note that the scheme is the "union" of the quasi-projective schemes $X_1 = P(x_0, x_1, x_2; \ ; x_1)$ and $X_2 = P(x_0, x_1, x_2; \ ; x_0x_2)$.

Now $X_1 = P(x_0, x_1, x_2; \ ; x_1) = Q(y_0, y_2; ;) = U_1$ is just affine 2-space $\mathbb{A}^2$. Here we think of $y_0 = x_0/x_1$ and $y_2 = x_2/x_1$.

Similarly $X_2 = P(x_0, x_1, x_2; \ ; x_0x_2) = Q(z_0, z_1; \ ; z_0)$ is naturally isomorphic to the affine scheme $A(z_0, z_1, u; uz_0 - 1) = U_2$. Here we think of $z_0 = x_0/x_2$, $z_1 = x_1/x_2$ and $u = x_2/x_0$.

Moreover, the intersection

$$X_1 \cap X_2 = P(x_0, x_1, x_2; \ ; x_0x_1x_2) = Q(w_0, w_1; \ ; w_0w_1) = V_{1,2}$$

is also the affine scheme $A(w_0, w_1, v; vw_0w_1 - 1)$. Here we think of $w_0 = x_0/x_2$, $w_1 = x_1/x_2$ and $v = x_2^2/(x_0x_1)$.

We see that $V_{1,2}$ sits inside U_2 by

$$(w_0, w_1, v) \mapsto (w_0, w_1, vw_1) = (z_0, z_1, u)$$

Similarly, $V_{1,2}$ sits inside U_1 by

$$(w_0, w_1, v) \mapsto (w_0^2 v, w_0 v) = (y_0, y_2)$$

One can also follow the "default" approach as given for a general quasi-projective scheme. However, that leads to many U_i's! So this geometric approach is simpler.