Schemes and Patching

Suppose that X is the scheme obtained by patching

- $U_1 = A(x, y, u; u(y - x^2) - 1)$ and
- $U_2 = A(x, y, v; v(y + x^2) - 1)$ along
- $U_3 = A(x, y, w; w(y^2 - x^4) - 1)$

which is an affine open subscheme in both of them in a natural way.

For the ring $R = \mathbb{Z}$, give an example

- a and b non-units in R
- $P = (c, d, e)$ in $U_1(R_a)$
- $Q = (f, g, h)$ in $U_2(R_b)$

So that these two points P and Q correspond to the same point in $U_3(R_{ab})$ and thus give a point in $X(R)$ which is *not* the image of a point in $U_1(R)$ or $U_2(R)$.

Solution

In U_1, we need $y - x^2$ to be a unit and in U_2 we need $y + x^2$ to be a unit on the overlap $y^2 - x^4$ is a unit. Thus, X is the open subscheme of $\mathbb{A}^2 = A(x, y;)$ which is the "complement" of $A(x, y; y - x^2, y + x^2)$.

A point of $X(\mathbb{Z})$ is a pair of integers (p, q) such that $q - p^2$ and $q + p^2$ generate the unit ideal. Since we do not want this to be in the image of $U_1(\mathbb{Z})$ or $U_2(\mathbb{Z})$, neither of $q \pm p^2$ should be a unit in $\mathbb{Z}$.

We can take $(p, q) = (2, 1)$ since $1 + 2^2 = 5$ and $1 - 2^2 = -3$; neither of these is a unit and they are co-prime.

We want $(2, 1)$ to be in $U_1(\mathbb{Z}_a)$. So we need $1 - 2^2$ to be invertible when we invert a. Thus, a is a multiple of 3.

We want $(2, 1)$ to be in $U_2(\mathbb{Z}_b)$. So we need $1 + 2^2$ to be invertible when we invert b. Thus, b is a multiple of 5.

We can take $a = 3$ and $b = 5$.

Now, the point $(c, d, e) = (2, 1, -1/3)$ is a point in $U_1(\mathbb{Z}_3)$ and $(f, g, h) = (2, 1, 1/5)$ is a point in $U_2(\mathbb{Z}_5)$.