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From where do we get the motivation of taking (Algebraic) spectrum of a ring, the collection of it's prime ideal?

$\boxed{k[x_1, \dots, x_p]} \xrightarrow{\phi} End(V)$ $\quad \bar{K}$

V is fin. dim'l $/K \supseteq k$

$K \cdot A =$ Image of ϕ is a finite dim'l K-algebra

$\Rightarrow A$ is Artinian & Noetherian.

$\Rightarrow m_1, \dots, m_n$ finitely many maximal ideals.

$m_1^{k_1} \cap \dots \cap m_n^{k_n} = (0)$

$A = A/m_1^{k_1} \times \dots \times A/m_n^{k_n}$

$k[x_1, \dots, x_p] \twoheadrightarrow KA \twoheadrightarrow KA/m_i =$ field ext of K $\Rightarrow \bar{K} = K$

Kernel is a maximal ideal

Image is a domain & so kernel is a prime ideal.

$k = \mathbb{F}_p$, or $\mathbb{Q}$. & $K = \mathbb{C}$ or $\bar{\mathbb{F}}_p$ some big field

$\Rightarrow$ $K \cdot A$ is

Kernel would only be a prime ideal

U open set (Subscheme) $\rightarrow$ shrinking

$\simeq Sp(R)$

$R_u \rightarrow R_{uv} \dots \dots$

"Big" multiplicatively closed sets $\quad S$

$S = R \setminus m$. m maximal

$S = R \setminus \mathfrak{p}$ $\quad \mathfrak{p}$ prime.

$S^{-1}R = R_{\mathfrak{p}}$.

~~Spec R~~ $\quad Spec(R_{\mathfrak{p}})$ unique maximal ideal

$m_{\mathfrak{p}} = \mathfrak{p}R_{\mathfrak{p}}$

$m_{\mathfrak{p}}$

Biplab Das, I unable to understand the concept of simultaneous eigenvalues.

Spectrum

Also please explain this.

More generally, if $R = \mathbb{Z}[x_1, \dots, x_p]/\langle f_1, \dots, f_q \rangle$ is a finitely presented ring then homomorphisms from it to a field F correspond to simultaneous eigenvalues for a collection of commuting operators that satisfy the given polynomials.

Simultaneous eigenvalues of the ring $R = \mathbb{Z}[x_1, \dots, x_p]/\langle f_1, \dots, f_q \rangle$ are points of the "spectrum" of this collection.

On the other hand, if $\mathbb{Z}[x_1, \dots, x_p] \to F$ is a homomorphism with $x_i \mapsto a_i$ where *at least one* of $f_j(\mathbf{a})$ is a *unit* then we see that $\mathbf{a}$ is *not* in the spectrum of R.

$\mathbb{Z}[x_1, x_2, \dots, x_p] \to F$

This motivates us to *define* the (algebraic) spectrum of R to be the collection of prime ideals in R.

The kernel of λ is a *prime* ideal in $\mathbb{Z}[x_1, \dots, x_p]$ since F is a domain.

Conversely, given a prime ideal $\mathfrak{p}$ in $\mathbb{Z}[x_1, \dots, x_p]$, the quotient ring is a domain.

Take $F_\mathfrak{p}$ to be the field of fractions of the domain $\mathbb{Z}[x_1, \dots, x_p]/\mathfrak{p}$.

We have a natural homomorphism $\lambda_\mathfrak{p} : \mathbb{Z}[x_1, \dots, x_p] \to F_\mathfrak{p}$.

Clearly, the x_i operates on the (one-dimensional) vector space $F_\mathfrak{p}$ via its image $a_i = \lambda_\mathfrak{p}(x_i)$ in $F_\mathfrak{p}$.

So we can think of $1 \in F_\mathfrak{p}$ as a "simultaneous eigenvector" of the operators x_i with the action $x_i \cdot 1 = a_i \cdot 1$.

1. What do you mean by operates?
2. Can you explain a little-bit about simultaneous eigenvector.

A, B $n \times n$ matrices

$A \cdot B = B \cdot A$

$A \cdot v = a \cdot v \quad a \in k.$

$A \cdot Bv = B \cdot A \cdot v = a\, Bv$ $\quad$ Bv is also an eigenvector with same eigenvalue

$\mathrm{Ker}(A - aI)$

B

w s.t. $(A - aI)w = 0$

& $Bw = b \cdot w$

$Aw = a \cdot w$

Commuting operators have simultaneous eigenvectors.

What contexts do AG in Alg. Geometry appear?

1. Solving equations.
2. V a vector space & $A_1, \dots, A_n$
   1 Collection of commuting elts of $\mathrm{End}(V)$

$\mathrm{Ker}\left(\boxed{\begin{array}{c} k[X_1, \dots, X_n] \to \mathrm{End}(V) \\ X_i \to A_i \end{array}}\right) = \text{Ideal}$

Can hope to use Alg. geom to study this.

Barnali: Why does this happen?

Open covers of a scheme

Given a scheme X, suppose $U_i \to X$ is a collection of open subschemes such that $U = \sqcup_i U_i \to X$ is a sheaf-theoretic surjection.

We say that such a collection is a *Zariski open cover* of X.

Let us clarify what this means in the case of an affine scheme $X = \mathrm{Sp}(R)$.

Since each U_i is an open subscheme of $\mathrm{Sp}(R)$, there is an ideal I_i in R such that $U_i = Q(R, I_i)$ is the scheme-theoretic complement of $\mathrm{Sp}(R/I_i)$.

Let J be the ideal in R generated by the ideals I_i as i varies.

If the ideal J is proper, then $\mathrm{Sp}(R/J) \to \mathrm{Sp}(R)$ is an element of $\mathrm{Sp}(R)(R/J)$. However the image of I_i in R/J is $\{0\}$ for all i. Thus, this element is *not* in the image of $\sqcup_i U_i$.

How it is the element of $\mathrm{Sp}(R)(R/J)$?

Closed sets

Shrunk closed annals

$\Downarrow$

prime ideals.