Are these definitions too complicated?

MTH437 — Introduction to Schemes

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Recall

The category **Scheme** of schemes was presented in two different ways.

1. As a subcategory of the category **Sheaf** (of functors from **CRing** to **Set** that satisfy the "Zariski patching condition" (also called the "Zariski (co-)sheaf conditions")).
2. As a subcategory of the category **LRSpace** of locally ringed spaces.

In both cases, this is a *full* subcategory; which means that morphisms of the "larger" category are morphisms of Schemes *without* any further restriction.

However, both cases leave us with very little "feeling" of the geometry of what we are studying.

Local vs. Global

The language of sheaves (as also the language of locally ringed spaces) provides the "book-keeping" in order to obtain "global" geometric objects from "local" objects.

In both cases, the local objects are associated with *Affine schemes*.

Hence, in order to properly understand the geometry of schemes, we *must* understand the geometry of affine schemes; i.e. *commutative algebra*.

However, the global aspects are *not* irrelevant. What we have called "book-keeping" is what leads, via the combinatorial study of patching, to algebraic topology (which includes cohomology).

In one sense, this is not very different from the situation in differential geometry, where analysis on $\mathbb{R}^n$ is the study of "local" questions.

Differential topology primarily deals with "global" questions that arise in manifolds. However, there *are* some global analytic effects of properties like compactness.

Affine schemes are intricate.

Unlike the study of manifolds, where the study of the unit ball around a point is 99% analysis and only 1% geometry (arising from differences of different dimensions), the geometry of affine schemes is quite intricate.

There is a lot of difference between the geometry of $X = A(x, y; x^2 + y^2 - 1)$ and $Y = A(x, y; x^2 + y^3 - 1)$ even though both are one dimensional.

In fact, one can show that *no* affine open subset X is isomorphic to an affine open subset of Y.

All the intricacy of global projective geometry is *already* present in affine spaces.

This can be seen via the "cone" construction and also via "Jouanolou's trick" which we will not discuss here.

Projective schemes arise in Commutative Algebra

In reverse, this also means that a *purely* algebraic approach to the study of commutative algebra will most likely prove inadequate.

Put differently, to understand *some* problems in commutative algebra we *must* "break it up" into open affine schemes and study questions (like cohomology) which arise via patching.

In summary, neither Commutative Algebra, nor Algebraic Geometry can be studied *without* studying the other!

In fact, one often needs to bring in Algebraic Topology as well, in order to understand "obstructions" to patching constructions.

This is (mistakenly) seen as a reason to declare that these are all too difficult to study.

Instead, one should think of it as an elephant which gives everyone something to study about it!

S-Schemes

Given a category $\mathcal{C}$ and an object S of $\mathcal{C}$, we define the "slice" category $\mathcal{C}/S$ as follows:

- Objects are morphisms $a_X : X \to S$.
- Morphisms $a_X \to a_Y$ are morphisms $f : X \to Y$ such that $a_Y \circ f = a_X$.

Given a scheme S, the slice category **Scheme**$/S$ is called the category of S-schemes.

When $S = \mathrm{Sp}(R)$, we sometimes also say that this is an R-scheme.

Note that there is a *unique* morphism $X \to \mathrm{Sp}(\mathbb{Z})$ for *any* scheme X. This uses the natural (and unique) ring homomorphism $\mathbb{Z} \to R$ for a ring R.

Thus, every scheme is a $\mathbb{Z}$-scheme.

We now look at one important *further* restriction on k-schemes for a fixed commutative ring k.

Every k scheme X is a quotient of a disjoint union $U = \sqcup_i U_i$ of affine schemes $U_i = \mathrm{Sp}(R_i)$ where R_i is a k-algebra.

We say X is a *k-scheme of finite type* if the above is a finite union and *each* R_i is a finitely generated k-algebra.

We have seen that a *finite* disjoint union of affine schemes is itself an affine scheme.

Thus, we can also say that a k-scheme of finite type is one that is the quotient of $U = \mathrm{Sp}(R)$ where R a finitely generated k-algebra.

In particular, if X is a quasi-projective k-scheme

$$X = Q_k(x_0, \ldots, x_p; f_1, \ldots, f_q; g_1, \ldots, g_r)$$

where f_i's and g_j's are homogeneous polynomials with coefficients in k, then X is a k-scheme of finite type.

One could say that the *primary* object of interest in algebraic geometry is the category $\mathbf{QProj}_k$ of quasi-projective k-schemes for various k.

For example k could be one of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{F}_p$, $\mathbb{R}$, $\mathbb{C}$ and so on.

Thus, it would seen reasonable to restrict one's attention to the category $\mathbf{FTScheme}_k$ of k-schemes of finite type.

One can ask if there are *simpler* approaches (avoiding sheaves!) to the study of such schemes.

Note that commutative algebra *does* involve questions about rings like $k(x)$ which are *not* finitely generated k-algebras.

Zariski-Weil foundations

One of the older approaches to defining the fundamental objects in algebraic geometry was the approach of Zariski, Weil and Chow.

Let us formulate what they were trying to do in the language of schemes.

Given a scheme X and a ring R, we have a set $X(R)$. Moreover, if $U \to X$ is an open subscheme, then $U(R)$ is a subset of $X(R)$.

Let us define a topology on $X(R)$ by declaring such subsets $U(R)$ as open. One checks easily that this makes $X(R)$ into a topological space.

Now, $X(R)$ may be "too small" for some rings R.

Hilbert's Nullstellensatz suggests that $X(K)$ is "big enough" for a "large enough" field (like $\mathbb{C}$).

For *reduced* schemes of finite type over k, the functor that sends X to $X(K)$ is *faithful*.

This means that if two morphisms $f, g : X \to Y$ are *equal* on K-points, then they are equal as morphisms.

Thus, we could ask if there is a nice geometric description of the image.

However, there are examples of morphism $X \to Y$ that are *not* isomorphisms that induce a homeomorphism $X(K) \to Y(K)$.

In the next lecture, we will look at a way to resolve this issue.