Locally Ringed Spaces

MTH437 — Introduction to Schemes

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18th November 2021

Recall

Given a scheme X, we produced a category $\mathcal{T}_X$ and a topological space $\sigma(X)$.

The category $\mathcal{T}_X$ is the same as the category of open sets in the topological space $\sigma(X)$. ("Same" means it is isomorphic in a natural way.)

An object F of **Sheaf** gives a sheaf on the category $\mathcal{T}_X$. Hence, we get a sheaf $\tilde{F}$ on $\sigma(X)$.

In particular, we have the sheaf $\mathcal{O}_X = \widetilde{\mathbb{A}^1}$ which is a sheaf of rings.

For a ring R we have a scheme $\mathrm{Sp}(R)$ and a topological space $\mathrm{Spec}(R)$ so that $\sigma(\mathrm{Sp}(R)) = \mathrm{Spec}(R)$.

Moreover, there is a basis for this topology made of open sets of the form $\mathrm{Spec}(R_u)$ as u varies over elements of R.

For every such open set we have a natural isomorphism $R_u \cong \mathcal{O}(\mathrm{Spec}(R_u))$.

A point $\mathfrak{p}$ of $\mathrm{Spec}(R)$ is associated with a prime ideal of R and open sets of the above form containing it are associated with u such that $u \notin \mathfrak{p}$.

Recall that the *stalk* of a sheaf $\mathcal{F}$ on $\mathcal{T}_X$ at a point $\mathfrak{p}$ is defined as the direct limit of $\mathcal{F}(U)$ as U runs over open sets containing $\mathfrak{p}$.

Hence, we see that the stalk of the sheaf $\mathcal{O}$ at a point $\mathfrak{p}$ of $\mathrm{Spec}(R)$ is the local ring $R_{\mathfrak{p}}$.

Given a scheme X we have a natural morphism $X \to \mathrm{Sp}(\mathcal{O}(X))$ and the scheme X is affine if and only if this is an isomorphism.

Given a scheme X and a point $\mathfrak{p}$ of $\sigma(X)$, there is an open subscheme U of X which is *affine* and $\mathfrak{p}$ is a point of $\sigma(U)$.

It follows that the stalk of $\mathcal{O}$ at $\mathfrak{p}$ is a local ring.

In fact, if $R = \mathcal{O}(U)$, then $\mathfrak{p}$ is associated with a prime ideal in R and the stalk of $\mathcal{O}$ at $\mathfrak{p}$ is $R_{\mathfrak{p}}$.

We use the notation $\mathcal{O}_{X,\mathfrak{p}}$ to denote this stalk.

Push forward sheaf

Given a continuous map $f : X \to Y$ of topological spaces, we get a natural functor f^{-1} from $\mathcal{T}_Y$ to $\mathcal{T}_X$ which sends an open set U in Y to the open set $f^{-1}(U)$.

Given a contravariant functor F from $\mathcal{T}_X$ to **Set** we can compose with the above functor to get a contravariant functor $f_*(F)$ from $\mathcal{T}_Y$ to **Set**.

One checks that if F is a sheaf, then so is $f_*(F)$. (*Hint*: the inverse image of an open cover of U is an open cover of $f^{-1}(U)$.)

Space of a sheaf

Given a sheaf F on $\mathcal{T}_X$, let $\sigma(F)$ denote the disjoint union $\sqcup_{x\in X} F_x$ of the stalks of F with the topology as given below.

Given an open set U of X, an element $s \in F(U)$ and a point x of U, we get a natural element s_x in the stalk F_x.

We define the basic open set $B(s, U)$ to consist of the subset $\{s_x : x \in U\}$ of $\sigma(F)$.

This gives a topological space $\sigma(F)$ with a natural map $\sigma(F) \to X$ (which sends s_x to x). Moreover, this map is a local homeomorphism.

Conversely, one can show that a local homeomorphism $Y \to X$ gives us a sheaf $\tilde{Y}$ on $\mathcal{T}_X$ called the sheaf of sections of $Y \to X$.

Pull-back sheaf

If $f : X \to Y$ is a continuous map and $S \to Y$ is a local homeomorphism, then one checks that $S \times_Y X \to X$ is *also* a local homeomorphism.

Starting with a sheaf F on Y, we get a local homeomorphism $\sigma(F) \to Y$ and then a local homeomorphism $\sigma(F) \times_Y X \to X$.

Note that, as a set, $\sigma(F) \times_Y X = \sqcup_{x \in X} F_{f(x)}$.

This is associated with the sheaf $\widetilde{\sigma(F) \times_Y X}$ on $\mathcal{T}_X$.

This sheaf on $\mathcal{T}_X$ is denoted as $f^{-1}(F)$ and called the pull-back sheaf.

One easily checks that the stalk of $f^{-1}(F)$ at x is the same as the stalk $F_{f(x)}$.

Adjunction for sheaves

Given a continuous map $f : X \to Y$, a sheaf F on X and a sheaf G on Y, there is a natural isomorphism

$$\mathrm{Mor}(f^{-1}G, F) = \mathrm{Mor}(G, f_*(F))$$

Here, the left-hand-side has morphisms of functors on $\mathcal{T}_X$ and the right-hand-side has morphisms of functors on $\mathcal{T}_Y$.

In fact, this property can be used to *define* f^{-1} given f_* and vice versa.

Morphisms of Schemes

Recall the σ is a functor from **Scheme** to **Top**.

In other words, given a morphism $f : X \to Y$ of schemes, the resulting map $\sigma(f) : \sigma(X) \to \sigma(Y)$ is continuous.

As seen earlier, "locally" each scheme is made up of affine schemes.

Similarly, each such morphism is "locally" associated with a morphism of affine schemes.

- Given $\mathfrak{p}$ in $\sigma(X)$ and $\mathfrak{q} = f(\mathfrak{p})$.
- There is an affine open subscheme $V \to Y$ such that $\mathfrak{q}$ lies in $\sigma(V)$.
- There is an affine open subscheme $U \to X$ such that $\mathfrak{p}$ lies in $\sigma(U)$.
- f restricts to a morphism $U \to V$ given by a ring homomorphism $\mathcal{O}(V) \to \mathcal{O}(U)$.

It follows that we have a ring homomorphism $\mathcal{O}_{Y,\mathfrak{q}} \to \mathcal{O}_{X,\mathfrak{p}}$.

Moreover, this is a *local ring homomorphism* in the sense that the inverse image of the maximal ideal in $\mathcal{O}_{X,\mathfrak{p}}$ is the maximal ideal in $\mathcal{O}_{Y,\mathfrak{q}}$.

This follows since $f(\mathfrak{p}) = \mathfrak{q}$.

These ring homomorphisms can be put together to give a morphism $f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$ of sheaves on X.

This morphism is made up of *ring homomorphisms*. (Note that both of these as sheaves of rings.)

Locally Ring Spaces

A locally ringed space is a topological X with a sheaf of rings $\mathcal{O}_X$ on it such that the stalk $\mathcal{O}_{X,\mathfrak{p}}$ at each point $\mathfrak{p}$ of X is a local ring.

A morphism of locally ring spaces is a continuous map $f : X \to Y$ and a homomorphism $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ of sheaves of rings such that at stalk level $\mathcal{O}_{Y,f(\mathfrak{p})} \to \mathcal{O}_{X,\mathfrak{p}}$ is a local ring homomorphism.

We denote by **LRSpace** the category of locally ringed spaces.

We have produced a functor **Scheme** to **LRSpace**. Let us denote this as Σ.

Note that $\mathrm{Mor}(X, Y) = \mathrm{Mor}(\Sigma(X), \Sigma(Y))$ for schemes X and Y since this is easily checked for affine schemes and then "patched".

Thus, we can think of **Scheme** as a subcategory of **LRSpace**.