Sheaves on schemes

MTH437 — Introduction to Schemes

Kapil Hari Paranjape

IISER Mohali

8th November 2021

Properties for morphisms

We extended the notion of open/closed subschemes to a subfunctor $Y \to X$ in **Fun**.

Given an element $a \in X(R)$ we defined Y_a as the *fibre product* $\mathrm{Sp}(R) \times_X Y$ of functors which is a subfunctor of $\mathrm{Sp}(R)$.

We say that $Y \to X$ is open (respectively closed) if Y_a is an open (respectively closed) subscheme of the affine scheme $\mathrm{Sp}(R)$ for *every* $a \in X(R)$ for *every* ring R.

Recall that $Y_a \to \mathrm{Sp}(R)$ is an open subscheme if there is an ideal I so that $Y_a \to \mathrm{Sp}(R)$ identifies Y_a with $Q(R, I)$, the quasi-affine scheme which is the scheme-theoretic complement of $\mathrm{Sp}(R/I)$ in $\mathrm{Sp}(R)$. Recall, that for a ring T

$$Q(R, I)(T) = \{f : R \to T | f(I)T = T\} \subset \mathrm{Hom}(R, T) = \mathrm{Sp}(R)(T)$$

Open covers of a scheme

Given a scheme X, suppose $U_i \to X$ is a collection of open subschemes such that $U = \sqcup_i U_i \to X$ is a sheaf-theoretic surjection.

We say that such a collection is a *Zariski open cover* of X.

Let us clarify what this means in the case of an affine scheme $X = \mathrm{Sp}(R)$.

Since each U_i is an open subscheme of $\mathrm{Sp}(R)$, there is an ideal I_i in R such that $U_i = Q(R, I_i)$ is the scheme-theoretic complement of $\mathrm{Sp}(R/I_i)$.

Let J be the ideal in R generated by the ideals I_i as i varies.

If the ideal J is proper, then $\mathrm{Sp}(R/J) \to \mathrm{Sp}(R)$ is an element of $\mathrm{Sp}(R)(R/J)$. However the image of I_i in R/J is $\{0\}$ for all i. Thus, this element is *not* in the image of $\sqcup_i U_i$.

Thus $\sum_i I_i = J = R$ and so we can find finitely many $u_t \in I_{i_t}$, for $t = 1, \ldots, k$ which generate the unit ideal.

> *For $t = 1, \ldots, k$ there are elements u_t in R such that $\mathrm{Sp}(R_{u_t}) \to U_{i_t} \to \mathrm{Sp}(R)$ is a factoring and $\langle u_1, \ldots, u_k \rangle = R$.*

In other words, there is a *refinement* of the cover by the affine open subschemes $\mathrm{Sp}(R_{u_t})$.

Sheaf property in terms of open covers

Suppose we are given a Zariski open cover $(U_i \to X)$ as above.

Given a functor F from **CRing** to **Set**, and a morphism $f : X \to F$, we have, by composition, morphism $f_i : U_i \to F$.

Recall that $U_i \times_X U_j$ represents the intersection of U_i and U_j in X. Thus, we see that f_i and f_j restrict to the same element on $U_i \times_X U_j$.

> **Claim**: *If* F is in **Sheaf**, then given $f_i : U_i \to F$ for each i such that f_i and f_j restrict to the *same* morphism on $U_i \times_X U_j$, there is a *unique* morphism $f : X \to F$ which gives f_i on restriction to U_i.

Let us indicate a proof of this claim.

To produce a morphism $X \to F$, we need to show that for each ring R, there is a natural map $X(R) \to F(R)$.

By the Yoneda Lemma, for any functor G, the set $G(R)$ is identified with morphisms $\mathrm{Sp}(R) \to G$.

So, given a morphism $\mathrm{Sp}(R) \to X$ we need to produce a morphism $\mathrm{Sp}(R) \to F$.

Let us fix such a morphism $\mathrm{Sp}(R) \to X$.

We note that $(V_i = U_i \times_X \mathrm{Sp}(R) \to \mathrm{Sp}(R))$ is an open cover of $\mathrm{Sp}(R)$.

As seen above, there is a refinement $\mathrm{Sp}(R_{u_t}) \to V_{i_t} \to \mathrm{Sp}(R)$, where $u_1, \dots, u_k$ generate the unit ideal on R.

By restriction of f_i from U_i, we get $f_t : \mathrm{Sp}(R_{u_{i_t}}) \to F$.

Moreover, f_t and f_s are the restrictions of f_{i_t} and f_{i_s}. Hence, they restrict to the same element on $\mathrm{Sp}(R_{u_{i_t} u_{i_s}})$ from $U_{i_t} \times_X U_{i_s}$.

By the sheaf property of F, we get a morphism $\mathrm{Sp}(R) \to F$ as required.

"Classical" Sheaf Theory

Given a topological space X, we have a category $\mathcal{T}_X$ whose objects are open subsets U of X and morphisms $i_V^U : V \to U$ are the natural inclusions of open subsets in one another *as* subsets of X.

We see that there is *at most* one morphism $V \to U$ for any pair of objects in $\mathcal{T}_X$.

Moreover, X is an object of $\mathcal{T}_X$ and there is a *unique* morphism $U \to X$ for every object of $\mathcal{T}_X$.

Given a *contravariant* functor F from $\mathcal{T}_X$ to **Set** and an element a in $F(U)$, we get an element $a_V = F(i_V^U)(a)$ in $F(V)$ for every open subset V of U.

Conversely, given an *open cover* (V_j) of U and elements a_j in $F(V_j)$ for each j, we can ask for a condition to get an element a of $F(U)$ such that $F(i^U_{V_j})(a) = a_j$.

Clearly, a necessary condition is that

$$F(i^{V_j}_{V_j \cap V_k})(a_j) = F(i^{V_k}_{V_j \cap V_k})(a_k) \text{ in } F(V_j \cap V_k)$$

If this condition is sufficient for *all* open covers (V_i) of *all* open sets U, we say that F is a *sheaf* (in the classical sense) on $\mathcal{T}_X$.

The category $\mathcal{T}_X$

We now extend this idea to a scheme X.

The category $\mathcal{T}_X$ is defined as follows:

- The objects of the category $\mathcal{T}_X$ are open subfunctors $i_U : U \to X$.
- The morphisms of the category are morphisms $i_V^U : V \to U$ such that $i_U \circ i_V^U = i_V$.

Note that *if* there is a morphism i_V^U, then it is uniquely determined by i_U and i_V.

We see that this category is *similar* to the category of open subsets of a topological space.

In fact, it is not difficult to show that this is the same as the category $\mathcal{T}_{\sigma(X)}$ associated to the topological space $\sigma(X)$.

Given F an object of **Sheaf**, we define, for $U \to X$ in $\mathcal{T}_X$, the set $\tilde{F}(U) = \mathrm{Mor}(U, F)$.

By the previous discussion, we see that $\tilde{F}$ is a sheaf in the classical sense on $\mathcal{T}_X$.

Equivalently, we can think of $\tilde{F}$ as a sheaf on $\mathcal{T}_{\sigma(X)}$.

More generally, a classical sheaf on $\mathcal{T}_X$ is called a (Zariski) sheaf on X by abuse of terminology.

Sheaves with structure

One important example of a sheaf on X is the sheaf $\mathcal{O}$ which associates $\mathcal{O}(U)$ to each open set $U \to X$. (Recall that $\mathcal{O}(U) = \mathrm{Mor}(U, \mathbb{A}^1)$; so $\mathcal{O} = \widetilde{\mathbb{A}^1}$ with notation as above.)

Sometimes we write this as $\mathcal{O}_X$ to emphasize that we are thinking of the sheaf on X.

This is not just a sheaf of sets but a sheaf of *rings*. In other words, we have a contravariant functor $\mathcal{T}_X$ to the category **Ring** of rings which has the sheaf property.

Similarly, a sheaf of abelian groups on X is a contravariant functor $\mathcal{T}_X$ to the category **Ab** of abelian groups which has the sheaf property.

A homomorphism $M \to N$ of sheaves of abelian groups (or rings) is a morphism of sheaves for which the set map $M(U) \to N(U)$ is a homomorphism of abelian groups (respectively rings) for each $U \to X$.

Given a sheaf M of abelian groups on X, it is not difficult to see that the association of $\text{End}(M(U))$ to $U \to X$ gives a *sheaf* of rings on X. We denote this as $\text{End}(M)$.

Given a sheaf M of abelian groups and a sheaf R of rings on X, we can ask for a homomorphism $R \to \text{End}(M)$ of sheaves of rings.

Given such a homomorphism we say that M is called a sheaf of modules over the sheaf of rings R.

In the next lecture we will talk about a special class of sheaves of $\mathcal{O}_X$ modules.