Exercises:

5.1. Prove the following proposition, due to Kaplansky: Let Λ be a ring in which every left ideal is projective. Then every submodule of a free Λ-module is isomorphic to a direct sum of modules each of which is isomorphic to a left ideal in Λ. Hence every submodule of a projective module is projective. (Hint: Proceed as in the proof of Theorem 5.1.)

5.2. Prove that a submodule of a finitely-generated module over a principal ideal domain is finitely-generated. State the fundamental theorem for finitely-generated modules over principal ideal domains.

5.3. Let A, B, C be finitely generated modules over the principal ideal domain Λ. Show that if $A \oplus C \cong B \oplus C$, then $A \cong B$. Give counterexamples if one drops (a) the condition that the modules be finitely generated, (b) the condition that Λ is a principal ideal domain.

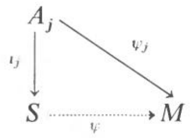
5.4. Show that submodules of projective modules need not be projective. ($\Lambda = \mathbb{Z}_{p^2}$, where p is a prime. $\mathbb{Z}_p \rightarrowtail \mathbb{Z}_{p^2} \twoheadrightarrow \mathbb{Z}_p$ is short exact but does not split!)

5.5. Develop a theory of linear transformations $T: V \to V$ of finite-dimensional vectorspaces over a field K by utilizing the fundamental theorem in the integral domain $K[T]$.

6. Dualization, Injective Modules

We introduce here the process of dualization only as a heuristic procedure. However, we shall see in Chapter II that it is a special case of a more general and canonical procedure. Suppose given a statement involving only modules and homomorphisms of modules; for example, the characterization of the direct sum of modules by its universal property given in Proposition 3.2:

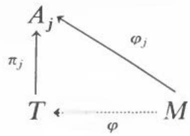
"The system consisting of the direct sum S of modules $\{A_j\}$, $j \in J$, together with the homomorphisms $\iota_j: A_j \to S$, is characterized by the following property. To any module M and homomorphisms $\{\psi_j: A_j \to M\}$, $j \in J$, there is a unique homomorphism $\psi: S \to M$ such that for every $j \in J$ the diagram



is commutative."

The *dual* of such a statement is obtained by "reversing the arrows"; more precisely, whenever in the original statement a homomorphism occurs we replace it by a homomorphism in the opposite direction. In our example the dual statement reads therefore as follows:

"Given a module T and homomorphisms $\{\pi_j: T \to A_j\}$, $j \in J$. To any module M and homomorphisms $\{\varphi_j: M \to A_j\}$, $j \in J$, there exists a unique homomorphism $\varphi: M \to T$ such that for every $j \in J$ the diagram



is commutative."

It is readily seen that this is the universal property characterizing the direct product of modules $\{A_j\}$, $j \in J$, the π_j being the canonical projections (Proposition 3.3). We therefore say that the notion of the direct product is *dual* to the notion of the direct sum.

Clearly to dualize a given statement we have to express it entirely in terms of modules and homomorphisms (not elements etc.). This can be done for a great many – though not all – of the basic notions introduced in Sections 1, ..., 5. In the remainder of this section we shall deal with a very important special case in greater detail: We define the class of injective modules by a property dual to the defining property of projective modules. Since in our original definition of projective modules the term „surjective" occurs, we first have to find a characterization of surjective homomorphisms in terms of modules and homomorphisms only. This is achieved by the following definition and Proposition 6.1.

Definition. A module homomorphism $\varepsilon: B \to C$ is *epimorphic* or an *epimorphism* if $\alpha_1 \varepsilon = \alpha_2 \varepsilon$ implies $\alpha_1 = \alpha_2$ for any two homomorphisms $\alpha_i: C \to M$, $i = 1, 2$.

Proposition 6.1. *$\varepsilon: B \to C$ is epimorphic if and only if it is surjective.*

Proof. Let $B \xrightarrow{\varepsilon} C \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} M$. If ε is surjective then clearly $\alpha_1 \varepsilon b = \alpha_2 \varepsilon b$ for all $b \in B$, implies $\alpha_1 c = \alpha_2 c$ for all $c \in C$. Conversely, suppose ε epimorphic and consider $B \xrightarrow{\varepsilon} C \underset{0}{\overset{\pi}{\rightrightarrows}} C/\varepsilon B$, where π is the canonical projection and 0 is the zero map. Since $0\varepsilon = 0 = \pi\varepsilon$, we obtain $0 = \pi$ and therefore $C/\varepsilon B = 0$ or $C = \varepsilon B$. □

Dualizing the above definition in the obvious way we have

Definition. The module homomorphism $\mu: A \to B$ is *monomorphic* or a *monomorphism* if $\mu\alpha_1 = \mu\alpha_2$ implies $\alpha_1 = \alpha_2$ for any two homomorphisms $\alpha_i: M \to A$, $i = 1, 2$.

Of course one expects that "monomorphic" means the same thing as "injective". For modules this is indeed the case; thus we have

Proposition 6.2. *$\mu: A \to B$ is monomorphic if and only if it is injective.*

Proof. If μ is injective, then $\mu\alpha_1 x = \mu\alpha_2 x$ for all $x \in M$ implies $\alpha_1 x = \alpha_2 x$ for all $x \in M$. Conversely, suppose μ monomorphic and $a_1, a_2 \in A$ such that $\mu a_1 = \mu a_2$. Choose $M = \Lambda$ and $\alpha_i: \Lambda \to A$ such that $\alpha_i(1) = a_i$, $i = 1, 2$. Then clearly $\mu\alpha_1 = \mu\alpha_2$; hence $\alpha_1 = \alpha_2$ and $a_1 = a_2$. □

It should be remarked here that from the categorical point of view (Chapter II) definitions should whenever possible be worded in terms of maps only. The basic notions therefore are "epimorphism" and "monomorphism", both of which are defined entirely in terms of maps. It is a fortunate coincidence that, for *modules*, "monomorphic" and "injective" on the one hand and "epimorphic" and "surjective" on the other hand mean the same thing. We shall see in Chapter II that in other categories monomorphisms do not have to be injective and epimorphisms do not have to be surjective. Notice that, to test whether a homomorphism is injective (surjective) one simply has to look at the homomorphism itself, whereas to test whether a homomorphism is monomorphic (epimorphic) one has, in principle, to consult all Λ-module homomorphisms.

We are now prepared to dualize the notion of a projective module.

Definition. A Λ-module I is called *injective* if for every homomorphism $\alpha: A \rightarrow I$ and every monomorphism $\mu: A \rightarrowtail B$ there exists a homomorphism $\beta: B \rightarrow I$ such that $\beta\mu = \alpha$, i.e. such that the diagram

$$\begin{array}{ccc} A & \overset{\mu}{\rightarrowtail} & B \\ {\scriptstyle\alpha}\downarrow & \swarrow {\scriptstyle\beta} & \\ I & & \end{array}$$

is commutative. Since μ may be regarded as an embedding, it is natural simply to say that I is injective if homomorphisms into I may be extended (from a given domain A to a larger domain B).

Clearly, one will expect that propositions about projective modules will dualize to propositions about injective modules. The reader must be warned, however, that even if the statement of a proposition is dualizable, the proof may *not* be. Thus it may happen that the dual of a true proposition turns out to be false. One must therefore give a proof of the dual proposition. One of the main objectives of Section 8 will, in fact, be to formulate and prove the dual of Theorem 4.7 (see Theorem 8.4). However, we shall need some preparation; first we state the dual of Proposition 4.5.

Proposition 6.3. *A direct product of modules* $\prod_{j \in J} I_j$ *is injective if and only if each* I_j *is injective.* □

The reader may check that in this particular instance the proof of Proposition 4.5 is dualizable. We therefore leave the details to the reader.

Exercises:

6.1. (a) Show that the zero module 0 is characterized by the property: To any module M there exists precisely one homomorphism $\varphi: 0 \rightarrow M$.
(b) Show that the dual property also characterizes the zero module.

6.2. Give a universal characterization of kernel and cokernel, and show that kernel and cokernel are dual notions.

6.3. Dualize the assertions of Lemma 1.1, the Five Lemma (Exercise 1.2) and those of Exercises 3.4 and 3.5.

6.4. Let $\varphi: A \to B$. Characterize $\operatorname{im}\varphi$, $\varphi^{-1} B_0$ for $B_0 \subseteq B$, without using elements. What are their duals? Hence (or otherwise) characterize exactness.

6.5. What is the dual of the canonical homomorphism $\sigma: \bigoplus_{i \in J} A_i \to \prod_{i \in J} A_i$? What is the dual of the assertion that σ is an injection? Is the dual true?

7. Injective Modules over a Principal Ideal Domain

Recall that by Corollary 5.2 every projective module over a principal ideal domain is free. It is reasonable to expect that the injective modules over a principal ideal domain also have a simple structure. We first define:

Definition. Let Λ be an integral domain. A Λ-module D is *divisible* if for every $d \in D$ and every $0 \neq \lambda \in \Lambda$ there exists $c \in D$ such that $\lambda c = d$. Note that we do not require the uniqueness of c.

We list a few examples:

(a) As $\mathbb{Z}$-module the additive group of the rationals $\mathbb{Q}$ is divisible. In this example c is uniquely determined.

(b) As $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ is divisible. Here c is *not* uniquely determined.

(c) The additive group of the reals $\mathbb{R}$, as well as $\mathbb{R}/\mathbb{Z}$, are divisible.

(d) A non-trivial finitely generated abelian group A is never divisible. Indeed, A is a direct sum of cyclic groups, which clearly are not divisible.

Theorem 7.1. *Let Λ be a principal ideal domain. A Λ-module is injective if and only if it is divisible.*

Proof. First suppose D is injective. Let $d \in D$ and $0 \neq \lambda \in \Lambda$. We have to show that there exists $c \in D$ such that $\lambda c = d$. Define $\alpha: \Lambda \to D$ by $\alpha(1) = d$ and $\mu: \Lambda \to \Lambda$ by $\mu(1) = \lambda$. Since Λ is an integral domain, $\mu(\xi) = \xi\lambda = 0$ if and only if $\xi = 0$. Hence μ is monomorphic. Since D is injective, there exists $\beta: \Lambda \to D$ such that $\beta\mu = \alpha$. We obtain

$$d = \alpha(1) = \beta\mu(1) = \beta(\lambda) = \lambda\beta(1).$$

Hence by setting $c = \beta(1)$ we obtain $d = \lambda c$. (Notice that so far no use is made of the fact that Λ is a principal ideal domain.)

Now suppose D is divisible. Consider the following diagram

$$\begin{array}{ccc} A & \overset{\mu}{\rightarrowtail} & B \\ {\scriptstyle\alpha}\downarrow & & \\ D & & \end{array}$$

We have to show the existence of $\beta: B \to D$ such that $\beta\mu = \alpha$. To simplify the notation we consider μ as an embedding of a submodule A into B. We look at pairs (A_j, α_j) with $A \subseteqq A_j \subseteqq B$, $\alpha_j: A_j \to D$ such that $\alpha_j|_A = \alpha$. Let Φ be the set of all such pairs. Clearly Φ is nonempty, since (A, α) is in Φ. The relation $(A_j, \alpha_j) \leqq (A_k, \alpha_k)$ if $A_j \subseteqq A_k$ and $\alpha_k|_{A_j} = \alpha_j$ defines an ordering in Φ. With this ordering Φ is inductive. Indeed, every chain (A_j, α_j), $j \in J$ has an upper bound, namely $(\bigcup A_j, \bigcup \alpha_j)$ where $\bigcup A_j$ is simply the union, and $\bigcup \alpha_j$ is defined as follows: If $a \in \bigcup A_j$, then $a \in A_k$ for some $k \in J$. We define $\bigcup \alpha_j(a) = \alpha_k(a)$. Plainly $\bigcup \alpha_j$ is well-defined and is a homomorphism, and

$$(A_j, \alpha_j) \leqq (\bigcup A_j, \bigcup \alpha_j).$$

By Zorn's Lemma there exists a maximal element $(\bar{A}, \bar{\alpha})$ in Φ. We shall show that $\bar{A} = B$, thus proving the theorem. Suppose $\bar{A} \neq B$; then there exists $b \in B$ with $b \notin \bar{A}$. The set of $\lambda \in \Lambda$ such that $\lambda b \in \bar{A}$ is readily seen to be an ideal of Λ. Since Λ is a principal ideal domain, this ideal is generated by one element, say λ_0. If $\lambda_0 \neq 0$, then we use the fact that D is divisible to find $c \in D$ such that $\bar{\alpha}(\lambda_0 b) = \lambda_0 c$. If $\lambda_0 = 0$, we choose an arbitrary c. The homomorphism $\bar{\alpha}$ may now be extended to the module $\tilde{A}$ generated by $\bar{A}$ and b, by setting $\tilde{\alpha}(\bar{a} + \lambda b) = \bar{\alpha}(\bar{a}) + \lambda c$. We have to check that this definition is consistent. If $\lambda b \in \bar{A}$, we have $\tilde{\alpha}(\lambda b) = \lambda c$. But $\lambda = \xi \lambda_0$ for some $\xi \in \Lambda$ and therefore $\lambda b = \xi \lambda_0 b$. Hence

$$\bar{\alpha}(\lambda b) = \bar{\alpha}(\xi \lambda_0 b) = \xi \bar{\alpha}(\lambda_0 b) = \xi \lambda_0 c = \lambda c.$$

Since $(\bar{A}, \bar{\alpha}) < (\tilde{A}, \tilde{\alpha})$, this contradicts the maximality of $(\bar{A}, \bar{\alpha})$, so that $\bar{A} = B$ as desired. □

Proposition 7.2. *Every quotient of a divisible module is divisible.*

Proof. Let $\varepsilon: D \twoheadrightarrow E$ be an epimorphism and let D be divisible. For $e \in E$ and $0 \neq \lambda \in \Lambda$ there exists $d \in D$ with $\varepsilon(d) = e$ and $d' \in D$ with $\lambda d' = d$. Setting $e' = \varepsilon(d')$ we have $\lambda e' = \lambda \varepsilon(d') = \varepsilon(\lambda d') = \varepsilon(d) = e$. □

As a corollary we obtain the dual of Corollary 5.3.

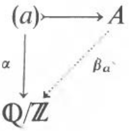
Corollary 7.3. *Let Λ be a principal ideal domain. Every quotient of an injective Λ-module is injective.* □

Next we restrict ourselves temporarily to abelian groups and prove in that special case

Proposition 7.4. *Every abelian group may be embedded in a divisible (hence injective) abelian group.*

The reader may compare this Proposition to Proposition 4.3, which says that every Λ-module is a quotient of a free, hence projective, Λ-module.

Proof. We shall define a monomorphism of the abelian group A into a direct product of copies of $\mathbb{Q}/\mathbb{Z}$. By Proposition 6.3 this will suffice. Let $0 \neq a \in A$ and let (a) denote the subgroup of A generated by a. Define $\alpha:(a) \to \mathbb{Q}/\mathbb{Z}$ as follows: If the order of $a \in A$ is infinite choose $0 \neq \alpha(a)$ arbitrary. If the order of $a \in A$ is finite, say n, choose $0 \neq \alpha(a)$ to have order dividing n. Since $\mathbb{Q}/\mathbb{Z}$ is injective, there exists a map $\beta_a: A \to \mathbb{Q}/\mathbb{Z}$ such that the diagram

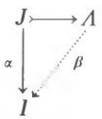


is commutative. By the universal property of the product, the β_a define a unique homomorphism $\beta: A \to \prod_{\substack{a \in A \\ a \neq 0}} (\mathbb{Q}/\mathbb{Z})_a$. Clearly β is a monomorphism since $\beta_a(a) \neq 0$ if $a \neq 0$. □

For abelian groups, the additive group of the integers $\mathbb{Z}$ is projective and has the property that to any abelian group $G \neq 0$ there exists a nonzero homomorphism $\varphi: \mathbb{Z} \to G$. The group $\mathbb{Q}/\mathbb{Z}$ has the dual properties; it is injective and to any abelian group $G \neq 0$ there is a nonzero homomorphism $\psi: G \to \mathbb{Q}/\mathbb{Z}$. Since a direct sum of copies of $\mathbb{Z}$ is called free, we shall term a direct product of copies of $\mathbb{Q}/\mathbb{Z}$ *cofree*. Note that the two properties of $\mathbb{Z}$ mentioned above do *not* characterize $\mathbb{Z}$ entirely. Therefore "cofree" is *not* the exact dual of "free", it is dual only in certain respects. In Section 8 the generalization of this concept to arbitrary rings is carried through.

Exercises:

7.1. Prove the following proposition: The Λ-module I is injective if and only if for every left ideal $J \subset \Lambda$ and for every Λ-module homomorphism $\alpha: J \to I$ the diagram



may be completed by a homomorphism $\beta: \Lambda \to I$ such that the resulting triangle is commutative. (Hint: Proceed as in the proof of Theorem 7.1.)

7.2. Let $0 \to R \to F \to A \to 0$ be a short exact sequence of abelian groups, with F free. By embedding F in a direct sum of copies of $\mathbb{Q}$, show how to embed A in a divisible group.

7.3. Show that every abelian group admits a unique maximal divisible subgroup.

7.4. Show that if A is a finite abelian group, then $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}) \cong A$. Deduce that if there is a short exact sequence $0 \to A' \to A \to A'' \to 0$ of abelian groups with A finite, then there is a short exact sequence $0 \to A'' \to A \to A' \to 0$.

7.5. Show that a torsion-free divisible group D is a $\mathbb{Q}$-vector space. Show that $\mathrm{Hom}_{\mathbb{Z}}(A, D)$ is then also divisible. Is this true for any divisible group D?

7.6. Show that $\mathbb{Q}$ is a direct summand in a direct product of copies of $\mathbb{Q}/\mathbb{Z}$.