Primes and Local Rings

MTH437 — Introduction to Schemes

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8th November 2021

Revision of Schemes

We are interested in "solving equations" (in commutative rings).

Given a system of equations X and a commutative ring R, the set of solutions can be denoted as $X(R)$.

Given a ring homomorphism $R \to S$, we can construct elements of $X(S)$ using the elements of $X(R)$.

This leads us to study functors **CRing** to **Set** as *objects* and natural transformations between them as *morphisms*. Let us denote this category as **Fun**.

Our primary interest is in functors that satisfy the (co-)sheaf condition; let us denote this subcategory of **Fun** as **Sheaf**.

Given a commutative ring R, we get a functor $\mathrm{Sp}(R)$ as follows.

- For commutative ring T, we have the set $\mathrm{Sp}(R)(T) = \mathrm{Hom}(R, T)$.
- Given a ring homomorphism $f : T \to S$, the set map $\mathrm{Sp}(R)(f) : \mathrm{Sp}(R)(T) \to \mathrm{Sp}(R)(S)$ is defined by composition of homomorphisms $g \mapsto f \circ g$.

$\mathrm{Sp}(R)$ is an object in **Sheaf**.

By Yoneda lemma, given an object X in **Fun** and a commutative ring R, we have an identification $F(R) = \mathrm{Mor}(\mathrm{Sp}(R), F)$.

In particular, we have an identification

$$\mathrm{Hom}(S, R) = \mathrm{Sp}(S)(R) = \mathrm{Mor}(\mathrm{Sp}(R), \mathrm{Sp}(S))$$

Thus, Sp is an identification of $\mathbf{CRing}^{\mathrm{opp}}$ with a sub-category **Affine** of **Sheaf**.

We denote $\mathbb{A}^1 = \mathrm{Sp}(\mathbb{Z}[x])$.

Note the natural identification $R = \mathrm{Hom}(\mathbb{Z}[x], R)$ gives $R = \mathrm{Mor}(\mathrm{Sp}(R), \mathbb{A}^1)$.

We have natural transformations $\mu, \alpha : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ given by addition and multiplication.

We also have natural $\mathbb{Z}$-points 0 and 1 in $\mathbb{A}^1$ by setting x to be 0 or 1 respectively.

For any object X of **Fun** this provides a *natural* ring structure on the set $\mathcal{O}(X) = \mathrm{Mor}(X, \mathbb{A}^1)$.

This can be thought of as the "ring of functions" on X.

Given X an object of **Fun** and a ring T we have $X(T) = \mathrm{Mor}(\mathrm{Sp}(T), X)$. Now, by composition

$$X(T) \times \mathcal{O}(X) = \mathrm{Mor}(\mathrm{Sp}(T), X) \times \mathrm{Mor}(X, \mathbb{A}^1) \to \mathrm{Mor}(\mathrm{Sp}(T), \mathbb{A}^1) = T$$

This gives, for each T-point of X a ring homomorphism $\mathcal{O}(X) \to T$ which is a T-point of $\mathrm{Sp}(\mathcal{O}(X))$.

We check that this is a morphism $X \to \mathrm{Sp}(\mathcal{O}(X))$.

We can say that X is an *affine* scheme if this map is an isomorphism.

We easily check that *any* morphism $X \to \mathrm{Sp}(T)$ *factors* as $X \to \mathrm{Sp}(\mathcal{O}(X)) \to \mathrm{Sp}(T)$.

A fancy way of saying this is that Sp and $\mathcal{O}$ are adjoint functors.

The contravariant functor Sp is from **CRing** to **Fun**.

The contravariant functor $\mathcal{O}$ is from **Fun** to **CRing**.

For a ring T in **CRing** and a functor F in **Fun** we have

$$\mathrm{Hom}(T, \mathcal{O}(X)) = \mathrm{Mor}(X, \mathrm{Sp}(T))$$

Since Sp identifies $\textbf{CRing}^{\mathrm{opp}}$ with a sub-category of **Sheaf**, we can say:

Algebraic Geometry contains Commutative Algebra!

Of course, it may also be said:

Algebraic Geometry is Commutative Algebra with the decorations provided by Category Theory!

Closed subschemes of affine schemes

Given a ring R and an ideal I in R, we have a natural homomorphism $R \to R/I$.

This gives a *subfunctor* $\mathrm{Sp}(R/I) \to \mathrm{Sp}(R)$. Such subfunctors are called "closed subschemes".

Given a ring T, the subset $\mathrm{Sp}(R/I)(T)$ consists of homomorphisms $f : R \to T$ where the image of I is the *zero* ideal in T.

Given a morphism $Y \to X$ of *affine schemes*, we see that Y is a closed subscheme of X if (and only if) the ring homorphism $\mathcal{O}(X) \to \mathcal{O}(Y)$ is *onto*.

Open subschemes of affine schemes

Open subschemes is the “complementary” notion to closed subschemes of affine schemes.

Given a ring R and an ideal I in R, we are looking for homomorphisms $f : R \to T$ where the image of I generates the unit ideal in T.

In other words, $f(I)T = T$.

We define $Q(R, I)(T)$ to be the subset of $\mathrm{Sp}(R)(T)$ that consists of such homomorphisms.

We say that such a functor $Q(R, I)$ is an *open subscheme* of $\mathrm{Sp}(R)$.

What is the relation of R to the functor $X = Q(R, I)$?

We have a morphism $X \to \mathrm{Sp}(S)$ where $S = \mathcal{O}(X)$.

Moreover, $X \to \mathrm{Sp}(R)$ gives us a morphism $\mathrm{Sp}(S) \to \mathrm{Sp}(R)$. Hence, a homomorphism $g : R \to S$. Let $J = g(I)S$.

A T-point of X gives a homomorphism $f : S \to T$ which by composition gives $f \circ g : R \to T$.

By definition of $X = Q(R, I)$, we see that $(f \circ g)(I)T = T$. Hence, $f(J)T = T$.

In summary:

> *A functor X is a quasi-affine scheme if (and only if) the morphism $X \to \mathrm{Sp}(\mathcal{O}(X))$ makes X an open subscheme.*

Warning

Note that if $X = Q(R, I)$, then it *need not* be the case that $\mathcal{O}(X) = R$!

For example, if $I = \langle g \rangle$, then $\mathcal{O}(X) = R_g$.

In this case, $X \to \mathrm{Sp}(R_g)$ is an isomorphism, which is also an open subscheme of itself!

Properties for morphisms

We now extend the notion of open/closed subschemes to a subfunctor $Y \to X$ in **Fun**.

For such a morphism given an element $a \in X(R)$ we define, the subfunctor $Y_a \to \mathrm{Sp}(R)$ as follows.

For a commutative ring T, the set $Y_a(T) \subset \mathrm{Sp}(R)(T)$ consists of those homomorphisms $f : R \to T$ such that image of a under $f : X(R) \to X(T)$ lies in the subset $Y(T)$.

Note: We can see that Y_a is the *fibre product* $\mathrm{Sp}(R) \times_X Y$ of functors.

We say that $Y \to X$ is open (respectively closed) if Y_a is an open (respectively closed) subscheme of the affine scheme $\mathrm{Sp}(R)$ for *every* $a \in X(R)$ for *every* ring R.

Schemes

A scheme X is the quotient sheaf functor of

$$E = \sqcup_{i,j} V_{i,j} \rightrightarrows U = \sqcup_i U_i$$

where:

- U_i are affine schemes,
- $\pi_1 : V_{i,j} \to U_i$ ($\pi_2 : V_{i,j} \to U_j$) is an open subscheme,
- $E(R)$ is an equivalence relation on $U(R) \times U(R)$.

Note that, unlike the previous definitions this is not *intrinsic*.

There are *many* possible choices of U_i's and $V_{i,j}$'s which give the *same* quotient sheaf functor.

The subcategory of **Sheaf** whose objects are schemes can be denoted by **Scheme**.

Using the fact that $V_{i,j} \to U_i$ is open, we can show that $U_i \to X$ is an open subfunctor.

We can thus characterise schemes as follows.

An object X of **Sheaf** is a scheme if there is a collection $(U_i \to X)_{i \in I}$ of open subfunctors of X such that U_i are affine and $\sqcup_i U_i \to X$ is a sheaf-theoretic quotient.

Intersections of open subschemes

As seen above, a scheme is locally made up of affine schemes like $\mathrm{Sp}(R)$.

If g and h are elements of R, then $\mathrm{Sp}(R_{gh})$ is the intersection of $\mathrm{Sp}(R_g)$ and $\mathrm{Sp}(R_h)$ in $\mathrm{Sp}(R)$.

How far can we "shrink" an affine scheme $\mathrm{Sp}(R)$ without making it empty?

A multiplicatively closed set S in R gives rise to a ring $S^{-1}R$:

- Elements are of the type (a, s) with a, s in R and $s \in S$.
- $(a, s) \sim (b, t)$ if there is a $u \in S$ such that $u(ta - sb) = 0$ in R.

Addition and multiplication are defined as usual.

One sees that $R \to S^{-1}R$ factors through R_s for $s \in S$, so $\mathrm{Sp}(S^{-1}R) \to \mathrm{Sp}(R)$ can be seen as the intersection of the open subschemes $\mathrm{Sp}(R_s)$.

As soon as S contains 0, we see that $S^{-1}R = \{0\}$ and so the intersection is empty.

So, we want to look at multiplicative subsets S in R such that $0 \notin S$.

Note that an ideal $\mathfrak{p}$ in R is *prime* if and only if its complement $S = R \setminus \mathfrak{p}$ is a multiplicatively closed subset.

This suggests that we look at rings of the form $R_{\mathfrak{p}} = (R \setminus \mathfrak{p})^{-1}R$ where $\mathfrak{p}$ is a prime ideal in R.

One checks that $m_{\mathfrak{p}} = \mathfrak{p}R_{\mathfrak{p}}$ is the unique maximal ideal in $R_{\mathfrak{p}}$.

So this is a typical example of a local ring.

Local Rings

Recall that a ring R with a unique maximal ideal $m \neq R$ is called a *local ring*. We often write this as a pair (R, m).

Given a local ring (R, m), every element of $R \setminus m$ is a unit in R and every element of m is *not* a unit in R.

It follows that, if $u_1, \ldots, u_k$ are elements of R that generate the unit ideal, then *at least* one of them is a unit in R.

This means that $R_{u_i} = R$ for at least one i.

Thus, the sheaf condition is a trivial condition for local rings.

In particular, if $F \to G$ is a sheaf-theoretic surjection $F(R) \to G(R)$ is onto for a local ring R.

Given a quasi-projective scheme $X = P(x_0, \ldots, x_p; f_1, \ldots, f_q; g_1, \ldots, g_r)$, we note that $X(R) = \tilde{X}(R)/E(R)$, where

$\tilde{X}(R)$ consists of tuples $\mathbf{a} = (a_0, \ldots, a_p) \in R^{p+1}$ such that:

- a_i is a unit for at least one i.
- $f_j(\mathbf{a}) = 0$ for all j, and
- $g_k(\mathbf{a})$ is a unit for at least one k.

... and $E(R)$ consists of pairs $(\mathbf{a}, \mathbf{b})$ such that there is a unit u in R such that

$$(a_0, \ldots, a_p) = (ub_0, \ldots, ub_p)$$

This is a special case of the following.

Suppose X is a scheme defined by patching

$$E = \sqcup_{i,j} V_{i,j} \rightrightarrows U = \sqcup_i U_i$$

where U_i and $V_{i,j}$ are affine aschemes.

Suppose (R, m) is a local ring. Then we see that $X(R) = U(R)/E(R)$.

Hence, we need not worry about the sheaf condition for X when R is a local ring.