Exercises:

4.1. Let V be a vector space of countable dimension over the field K. Let $\Lambda = \mathrm{Hom}_K(V, V)$. Show that, as K-vector spaces V is isomorphic to $V \oplus V$. We therefore obtain

$$\Lambda = \mathrm{Hom}_K(V, V) \cong \mathrm{Hom}_K(V \oplus V, V) \cong \mathrm{Hom}_K(V, V) \oplus \mathrm{Hom}_K(V, V) = \Lambda \oplus \Lambda .$$

Conclude that, in general, the free module on a set of n elements may be isomorphic to the free module on a set of m elements, with $n \neq m$.

4.2. Given two projective Λ-modules P, Q, show that there exists a *free* Λ-module R such that $P \oplus R \cong Q \oplus R$ is free. (Hint: Let $P \oplus P'$ and $Q \oplus Q'$ be free. Define $R = P' \oplus (Q \oplus Q') \oplus (P \oplus P') \oplus \cdots \cong Q' \oplus (P \oplus P') \oplus (Q \oplus Q') \oplus \cdots$.)

4.3. Show that $\mathbb{Q}$ is not a free $\mathbb{Z}$-module.

4.4. Need a direct product of projective modules be projective?

4.5. Show that if $0 \to N \to P \to A \to 0$, $0 \to M \to Q \to A \to 0$ are exact with P, Q projective, then $P \oplus M \cong Q \oplus N$. (Hint: Use Exercise 3.4.)

4.6. We say that A has a *finite presentation* if there is a short exact sequence $0 \to N \to P \to A \to 0$ with P finitely-generated projective and N finitely-generated. Show that

(i) if A has a finite presentation, then, for every exact sequence

$$0 \to R \to S \to A \to 0$$

with S finitely-generated, R is also finitely-generated;

(ii) if A has a finite presentation, it has a finite presentation with P free;

(iii) if A has a finite presentation every presentation $0 \to N \to P \to A \to 0$ with P projective, N finitely-generated is finite, and every presentation $0 \to N \to P \to A \to 0$ with P finitely-generated projective is finite:

(iv) if A has a presentation $0 \to N_1 \to P_1 \to A \to 0$ with P_1 finitely-generated projective, and a presentation $0 \to N_2 \to P_2 \to A \to 0$ with P_2 projective, N_2 finitely-generated, then A has a finite presentation (indeed, both the given presentations are finite).

4.7. Let $\Lambda = K(x_1, \ldots, x_n, \ldots)$ be the polynomial ring in countably many indeterminates $x_1, \ldots, x_n, \ldots$ over the field K. Show that the ideal I generated by $x_1, \ldots, x_n, \ldots$ is not finitely generated. Hence we may have a presentation $0 \to N \to P \to A \to 0$ with P finitely generated projective and N *not* finitely-generated.

5. Projective Modules over a Principal Ideal Domain

Here we shall prove a rather difficult theorem about principal ideal domains. We remark that a very simple proof is available if one is content to consider only finitely generated Λ-modules; then the theorem forms a part of the fundamental classical theorem on the structure of finitely generated modules over principal ideal domains.

Recall that a principal ideal domain Λ is a commutative ring without divisors of zero in which every ideal is principal, i.e. generated by one element. It follows that as a module every ideal in Λ is isomorphic to Λ itself.

Theorem 5.1. *Over a principal ideal domain Λ every submodule of a free Λ-module is free.*

Since projective modules are direct summands in free modules, this implies

Corollary 5.2. *Over a principal ideal domain, every projective module is free.*

Corollary 5.3. *Over a principal ideal domain, every submodule of a projective module is projective.*

Proof of Theorem 5.1. Let $P = \bigoplus_{j \in J} \Lambda_j$, where $\Lambda_j = \Lambda$, be a free module and let R be a submodule of P. We shall show that R has a basis. Assume J well-ordered and define for every $j \in J$ modules

$$\bar{P}_{(j)} = \bigoplus_{i<j} \Lambda_i, \qquad P_{(j)} = \bigoplus_{i \leqq j} \Lambda_i .$$

Then every element $a \in P_{(j)} \cap R$ may be written uniquely in the form (b, λ) where $b \in \bar{P}_{(j)}$ and $\lambda \in \Lambda_j$. We define a homomorphism $f_j : P_{(j)} \cap R \to \Lambda$ by $f_j(a) = \lambda$. Since the kernel of f_j is $\bar{P}_{(j)} \cap R$ we obtain an exact sequence

$$\bar{P}_{(j)} \cap R \rightarrowtail P_{(j)} \cap R \twoheadrightarrow \operatorname{im} f_j .$$

Clearly $\operatorname{im} f_j$ is an ideal in Λ. Since Λ is a principal ideal domain, this ideal is generated by one element, say λ_j. For $\lambda_j \neq 0$ we choose $c_j \in P_{(j)} \cap R$, such that $f_j(c_j) = \lambda_j$. Let $J' \subseteq J$ consist of those j such that $\lambda_j \neq 0$. We claim that the family $\{c_j\}$, $j \in J'$, is a basis of R.

First we show that $\{c_j\}, j \in J'$, is linearly independent. Let $\sum_{k=1}^{n} \mu_k c_{j_k} = 0$ and let $j_1 < j_2 < \cdots < j_n$. Then applying the homomorphism f_{j_n}, we get $\mu_n f_{j_n}(c_{j_n}) = \mu_n \lambda_{j_n} = 0$. Since $\lambda_{j_n} \neq 0$ this implies $\mu_n = 0$. The assertion then follows by induction on n.

Finally, we show that $\{c_j\}$, $j \in J'$, generates R. Assume the contrary. Then there is a least $i \in J$ such that there exists $a \in P_{(i)} \cap R$ which cannot be written as a linear combination of $\{c_j\}, j \in J'$. If $i \notin J'$, then $a \in \bar{P}_{(i)} \cap R$; but then there exists $k < i$ such that $a \in P_{(k)} \cap R$, contradicting the minimality of i. Thus $i \in J'$.

Consider $f_i(a) = \mu \lambda_i$ and form $b = a - \mu c_i$. Clearly

$$f_i(b) = f_i(a) - f_i(\mu c_i) = 0 .$$

Hence $b \in \bar{P}_{(i)} \cap R$, and b cannot be written as a linear combination of $\{c_j\}$, $j \in J'$. But there exists $k < i$ with $b \in P_{(k)} \cap R$, thus contradicting the minimality of i. Hence $\{c_j\}$, $j \in J'$, is a basis of E. □

Exercises:

5.1. Prove the following proposition, due to Kaplansky: Let Λ be a ring in which every left ideal is projective. Then every submodule of a free Λ-module is isomorphic to a direct sum of modules each of which is isomorphic to a left ideal in Λ. Hence every submodule of a projective module is projective. (Hint: Proceed as in the proof of Theorem 5.1.)

5.2. Prove that a submodule of a finitely-generated module over a principal ideal domain is finitely-generated. State the fundamental theorem for finitely-generated modules over principal ideal domains.

5.3. Let A, B, C be finitely generated modules over the principal ideal domain Λ. Show that if $A \oplus C \cong B \oplus C$, then $A \cong B$. Give counterexamples if one drops (a) the condition that the modules be finitely generated, (b) the condition that Λ is a principal ideal domain.

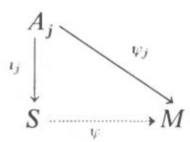
5.4. Show that submodules of projective modules need not be projective. ($\Lambda = \mathbb{Z}_{p^2}$, where p is a prime. $\mathbb{Z}_p \rightarrowtail \mathbb{Z}_{p^2} \twoheadrightarrow \mathbb{Z}_p$ is short exact but does not split!)

5.5. Develop a theory of linear transformations $T: V \to V$ of finite-dimensional vectorspaces over a field K by utilizing the fundamental theorem in the integral domain $K[T]$.

6. Dualization, Injective Modules

We introduce here the process of dualization only as a heuristic procedure. However, we shall see in Chapter II that it is a special case of a more general and canonical procedure. Suppose given a statement involving only modules and homomorphisms of modules; for example, the characterization of the direct sum of modules by its universal property given in Proposition 3.2:

"The system consisting of the direct sum S of modules $\{A_j\}$, $j \in J$, together with the homomorphisms $\iota_j: A_j \to S$, is characterized by the following property. To any module M and homomorphisms $\{\psi_j: A_j \to M\}$, $j \in J$, there is a unique homomorphism $\psi: S \to M$ such that for every $j \in J$ the diagram



is commutative."

The *dual* of such a statement is obtained by "reversing the arrows"; more precisely, whenever in the original statement a homomorphism occurs we replace it by a homomorphism in the opposite direction. In our example the dual statement reads therefore as follows:

"Given a module T and homomorphisms $\{\pi_j: T \to A_j\}$, $j \in J$. To any module M and homomorphisms $\{\varphi_j: M \to A_j\}$, $j \in J$, there exists a