Topological space of a Scheme

MTH437 — Introduction to Schemes

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Simultaneous eigenvalues

Given a commuting collection $a_1, \ldots, a_p$ of operators on a vector space V over a field F.

This amounts to giving a homomorphism $\phi : \mathbb{Z}[x_1, \ldots, x_p] \to \mathrm{End}_F(V)$ of rings defined by $x_i \mapsto a_i$ for $i = 1, \ldots, k$.

Note that $\mathrm{End}_F(V)$ is not necessarily commutative, but the image of a commutative ring under a homomorphism is commutative.

A *simultaneous* eigenvector of these operators is a non-zero vector $v \in V$ such that $a_i v = \lambda_i v$ with $\lambda_i \in F$ for $i = 1, \ldots, k$.

In particular, this corresponds to a homomorphism $\lambda : \mathbb{Z}[x_1, \ldots, x_p] \to F$ given by $x_i \mapsto \lambda_i$.

The kernel of λ is a *prime* ideal in $\mathbb{Z}[x_1, \ldots, x_p]$ since F is a domain.

Conversely, given a prime ideal $\mathfrak{p}$ in $\mathbb{Z}[x_1, \ldots, x_p]$, the quotient ring is a domain.

Take $F_{\mathfrak{p}}$ to be the field of fractions of the domain $\mathbb{Z}[x_1, \ldots, x_p]/\mathfrak{p}$.

We have a natural homomorphism $\lambda_{\mathfrak{p}} : \mathbb{Z}[x_1, \ldots, x_p] \to F_{\mathfrak{p}}$.

Clearly, the x_i operates on the (one-dimensional) vector space $F_{\mathfrak{p}}$ via its image $a_i = \lambda_{\mathfrak{p}}(x_i)$ in $F_{\mathfrak{p}}$.

So we can think of $1 \in F_{\mathfrak{p}}$ as a "simultaneous eigenvector" of the operators x_i with the action $x_i \cdot 1 = a_i \cdot 1$.

Spectrum

More generally, if $R = \mathbb{Z}[x_1, \ldots, x_p]/\langle f_1, \ldots, f_q \rangle$ is a finitely presented ring, then homomorphisms from it to a field F correspond to simultaneous eigenvalues for a collection of commuting operators that satisfy the given polynomials.

Simultaneous eigenvalues of the ring $R = \mathbb{Z}[x_1, \ldots, x_p]/\langle f_1, \ldots, f_q \rangle$ are points of the "spectrum" of this collection.

On the other hand, if $\mathbb{Z}[x_1, \ldots, x_p] \to F$ is a homomorphism with $x_i \mapsto a_i$ where *at least one* of $f_j(\mathbf{a})$ is a *unit* then we see that $\mathbf{a}$ is *not* in the spectrum of R.

This motivates us to *define* the (algebraic) spectrum of R to be the collection of prime ideals in R.

For a commutative ring R, we denote by $\mathrm{Spec}(R)$, the collection of prime ideals in R.

If $f : R \to S$ is a ring homomorphism, and $\mathfrak{p}$ is a prime ideal in S, then $S/\mathfrak{p}$ is a domain.

Thus, the kernel $\mathfrak{q} = f^{-1}(\mathfrak{p})$ of $R \to S \to S/\mathfrak{p}$ is *also* a prime ideal in R.

This gives a set map $f^* : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ which sends $\mathfrak{p}$ to $f^{-1}(\mathfrak{p})$.

Hence, Spec defines a *contravariant* functor **CRing** to **Set**.

Note that $\{0\}$ has *no* prime ideal so $\mathrm{Spec}(\{0\})$ is the empty set.

If g is an element of R, then prime ideals in R_g can be identified with prime ideals in R that *do not contain* g.

This gives an inclusion $\mathrm{Spec}(R_g) \subset \mathrm{Spec}(R)$ via the natural map $R \to R_g$.

If I is an ideal in R, the prime ideals in R/I can be identified with prime ideals in R that contain I.

This gives an inclusion $\mathrm{Spec}(R/I) \subset \mathrm{Spec}(R)$ via the natural map $R \to R/I$.

Zariski Topology

Let us define a subset of $\text{Spec}(R)$ of the form $\text{Spec}(R/I)$ to be a *closed* subset.

Given ideals I and J, one sees that if a prime $\mathfrak{p}$ contains $I \cap J$, then

- Either $\mathfrak{p}$ contains I, or
- There is an element $a \in I \setminus \mathfrak{p}$, and then for every $b \in J$, $ab \in I \cap J \subset \mathfrak{p}$ so that $b \in \mathfrak{p}$.

It follows that $\text{Spec}(R/I) \cup \text{Spec}(R/J) = \text{Spec}(R/(I \cap J))$.

In other words, the union of closed subsets in $\text{Spec}(R)$ is closed. as required.

Suppose that (I_α) is a collection of ideals in R.

A prime ideal $\mathfrak{p}$ contains *each* I_α if and only if $\mathfrak{p}$ contains $J = \sum_\alpha I_\alpha$.

In other words, the intersection of the closed sets $\mathrm{Spec}(R/I_\alpha)$ in $\mathrm{Spec}(R)$ is the closed set $\mathrm{Spec}(R/J)$, as required.

Thus, this definition of closed sets defines a topology on $\mathrm{Spec}(R)$ called the *Zariski Topology* on R.

We use the notation $\mathrm{Spec}(R)$ to refer to this topological space by abuse of notation.

Open sets

If g is an element of R, then $\text{Spec}(R_g) \subset \text{Spec}(R)$ is the complement of the closed set $\text{Spec}(R/\langle g \rangle)$.

Thus, it $\text{Spec}(R_g)$ is an *open* subset of $\text{Spec}(R)$.

We now see that sets of this form are a "basis" for open sets in the sense that every open set is a union of these.

More generally, if I is an ideal in R, then $\text{Spec}(R) \setminus \text{Spec}(R/I)$ is the collection of prime ideals that *do not contain* I.

For each such prime ideal $\mathfrak{p}$, there is an element g in $I \setminus \mathfrak{p}$.

It follows that $\text{Spec}(R) \setminus \text{Spec}(R/I)$ is the *union* of the (basic) open sets $\text{Spec}(R_g)$ as g varies over elements of $I \setminus \mathfrak{p}$.

Homomorphisms and continuity

Given a ring homomorphism $f : R \to S$, we have seen that there is a map $f^* : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ which sends a prime $\mathfrak{p}$ in S to $f^{-1}(\mathfrak{p})$ in R.

We now check that this is continuous.

Given $\mathrm{Spec}(R/I)$ a closed set in $\mathrm{Spec}(R)$. The prime $f^{-1}(\mathfrak{p})$ lies in $\mathrm{Spec}(R/I)$ if $I \subset f^{-1}(\mathfrak{p})$.

This is equivalent to saying the $f(I) \subset \mathfrak{p}$. Which is equivalent to saying theat $f(I)S \subset \mathfrak{p}$, since the latter is an ideal in S.

Thus, $f^*(\mathfrak{p})$ lies in $\mathrm{Spec}(R/I)$ if and only if $\mathfrak{p}$ lies in $\mathrm{Spec}(S/f(I)S$.

In other words, $(f^*)^{-1}(\mathrm{Spec}(R/I)) = \mathrm{Spec}(S/f(I)S)$ which is a closed set in $\mathrm{Spec}(S)$.

The functor σ

Note that if $U = \mathsf{Sp}(R)$ is the (affine scheme) functor **CRing** to **Set** which sends a ring T to the set $\mathrm{Hom}(R, T)$, then

$$R = \mathcal{O}(U) = \mathrm{Hom}(U, \mathbb{A}^1)$$

For an affine scheme U we define $\sigma(U) = \mathsf{Spec}(\mathcal{O}(U))$.

If $U = \mathsf{Sp}(R)$ and $V = \mathsf{Sp}(S)$, then morphisms (natural transformations) $V \to U$ are identified with ring homomorphisms $R \to S$.

We have shown that $f^* : \sigma(V) = \mathsf{Spec}(S) \to \sigma(U) = \mathsf{Spec}(R)$ is continuous.

Thus, we have a functor σ from affine schemes to topological spaces.

We now extend this functor to other schemes.

Extension to separated Schemes

Suppose X is a separated scheme defined by patching

$$E = \sqcup_{i,j} V_{i,j} \ \rightrightarrows \ U = \sqcup_i U_i$$

where U_i are affine schemes and $V_{i,j} \subset U_i \times U_j$ a closed subscheme (and thus affine) such that the projection maps $V_{i,j} \to U_i$ are open subschemes.

We then get maps of topological spaces

$$E' = \sqcup_{i,j} \sigma(V^{(i,j)}) \rightrightarrows U' \sqcup_i \sigma(U^{(i)})$$

Moreover, we claim that these maps define an *open equivalence relation* on the topological space U'.

- In other words, E' is a *subset* of U' and defines an equivalence relation on it.
- The map $\pi_1 : E' \to U'$ is an *open continuous map*.

It then follows that there is a natural *quotient topology* on U'/E'.

To prove the claim, we note that the morphism of affine schemes $V_{i,j} \to U_i$ is makes the former an open subscheme.

From the previous discussion, this means that $\sigma(V_{i,j}) \to \sigma(U_i)$ is an *open subset*.

In particular, this is an open map. Hence, $\pi_1 : E' \to U'$ is open and continuous.

Moreover, $\sigma(V_{i,j}) \to \sigma(U_i) \times \sigma(U_j)$ is an inclusion. Thus, $E' \to U' \times U'$ is an inclusion.

The fact that $E \to U \times U$ is an equivalence relation implies that $E' \to U' \times U'$ is also an equivalence relation. (As usual, one only needs to check transitivity.)

The claim thus follows.

We define $\sigma(X)$ to be this topological space associated with a scheme X.

Continuity of morphisms

Every point of $\sigma(X)$ lies in an open set of the form $\text{Spec}(R^{(i)})$ for some i in this definition.

Note that continuity of maps is a *local* property.

It follows that a morphism $X \to Y$ of separated schemes gives a continuous map $\sigma(X) \to \sigma(Y)$ of the underlying topological spaces.

Thus σ is a functor **SScheme** to **Top** where the former is the category of separated schemes and the latter is the category of topological spaces.

General schemes

We now expand the definition of σ to *all* schemes—not necessarily separated.

Such a scheme X can again be described by patching

$$E = \sqcup_{i,j} V_{i,j} \rightrightarrows U = \sqcup_i U_i$$

where U_i are affine schemes and $V_{i,j} \subset U_i \times U_j$ a subscheme (not necessarily closed) such that the projection maps $V_{i,j} \to U_i$ are open subschemes.

This means that $V_{i,j}$ are *quasi*-affine schemes.

Once we extend the definition of σ to such schemes, the rest of the argument above can be repeated in this case as well.

σ for Quasi-affine schemes

Given a commutative ring R and an ideal I, we defined the quasi-affine scheme $Q(R, I)$ as a functor **CRing** to **Set**.

The set $Q(R, I)(T)$ consists of homomorphisms $R \to T$ where the image of I generates the unit ideal in T.

As seen earlier, we can write $Q(R, I)$ as the sheaf-theoretic union of the affine subschemes $\mathrm{Sp}(R_g)$ as g varies over elements of I.

In particular, we note that $\sigma(Q(R, I))$ is isomorphic to the open subspace $\mathrm{Spec}(R) \setminus \mathrm{Spec}(R/I)$ of $\mathrm{Spec}(R)$.

Using this one can extend σ as a functor **Scheme** to **Set** by writing schemes of quotients of quasi-affine schemes.