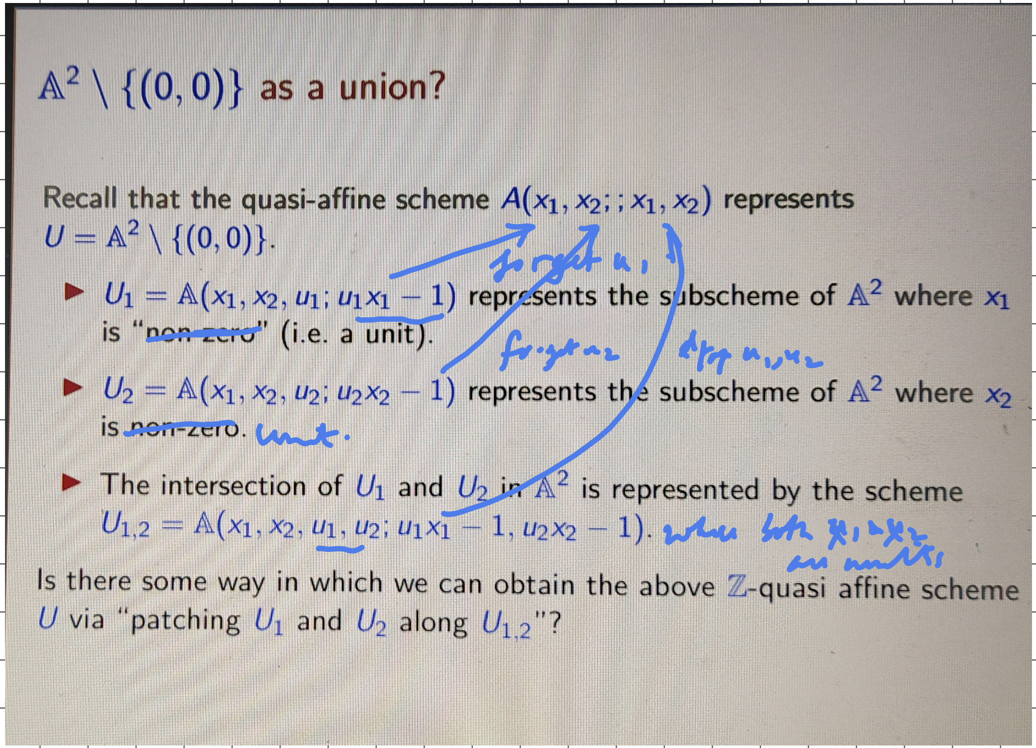
Prakash



$\mathbb{A}^2 \setminus \{(0,0)\}$ as a union?

Recall that the quasi-affine scheme $A(x_1, x_2; ; x_1, x_2)$ represents $U = \mathbb{A}^2 \setminus \{(0,0)\}$.

- $U_1 = \mathbb{A}(x_1, x_2, u_1; \underline{u_1 x_1 - 1})$ represents the subscheme of $\mathbb{A}^2$ where x_1 is "~~non-zero~~" (i.e. a unit).
- $U_2 = \mathbb{A}(x_1, x_2, u_2; u_2 x_2 - 1)$ represents the subscheme of $\mathbb{A}^2$ where x_2 is ~~non-zero~~. unit.
- The intersection of U_1 and U_2 in $\mathbb{A}^2$ is represented by the scheme $U_{1,2} = \mathbb{A}(x_1, x_2, \underline{u_1}, u_2; u_1 x_1 - 1, u_2 x_2 - 1)$. when both x_1 & x_2 are units

Is there some way in which we can obtain the above $\mathbb{Z}$-quasi affine scheme U via "patching U_1 and U_2 along $U_{1,2}$"?

forget u_1

forget u_2

drop u_1, u_2

1) In 3rd slide it is considered that U1 is subscheme of A^2 but it is not seems me obvious also there was no explanation for it . what i was thinking initially that it should be subscheme of A^4 than defining intersection and other stuffs will be more understandable.

one another thing which i could understand is that we can convert this affine scheme U1 to quasi affine scheme (x1,x2;;x1) . But exactly what you want to mean by saying U1 as subscheme of A2 it is not clear to me.

$U_1(R) \subset \underline{\mathbb{A}^2(R)}$ $\overset{R^2}{\|}$

$a_1, a_2, \boxed{b_1} ; a_1 b_1 = 1$ in R

$\overset{\uparrow}{R}\ \overset{\uparrow}{R}\ \overset{\uparrow}{R}$

a_1, b_2, c_1

$a_1 c_1 = 1$

$b_2 = c_1$

$U_2(R) \subset \mathbb{A}^2(R)$

(a_1, a_2) when a_2 is a unit.

$U_{1,2}(R) \subset \mathbb{A}^2(R) = R^2$

(a_1, a_2) $a_1 a_2$ is a unit.

$Q = A(x_1, x_2; ; x_1, x_2)$

$Q(R) = \{\underbrace{(a_1, a_2)}_{\in R^2} \mid \langle a_1, a_2 \rangle_R = R\}$

Not nec that a_1 or a_2 be a unit!

$\underline{U_1(R) \cup U_2(R) \neq Q(R)}$ but possible by patch

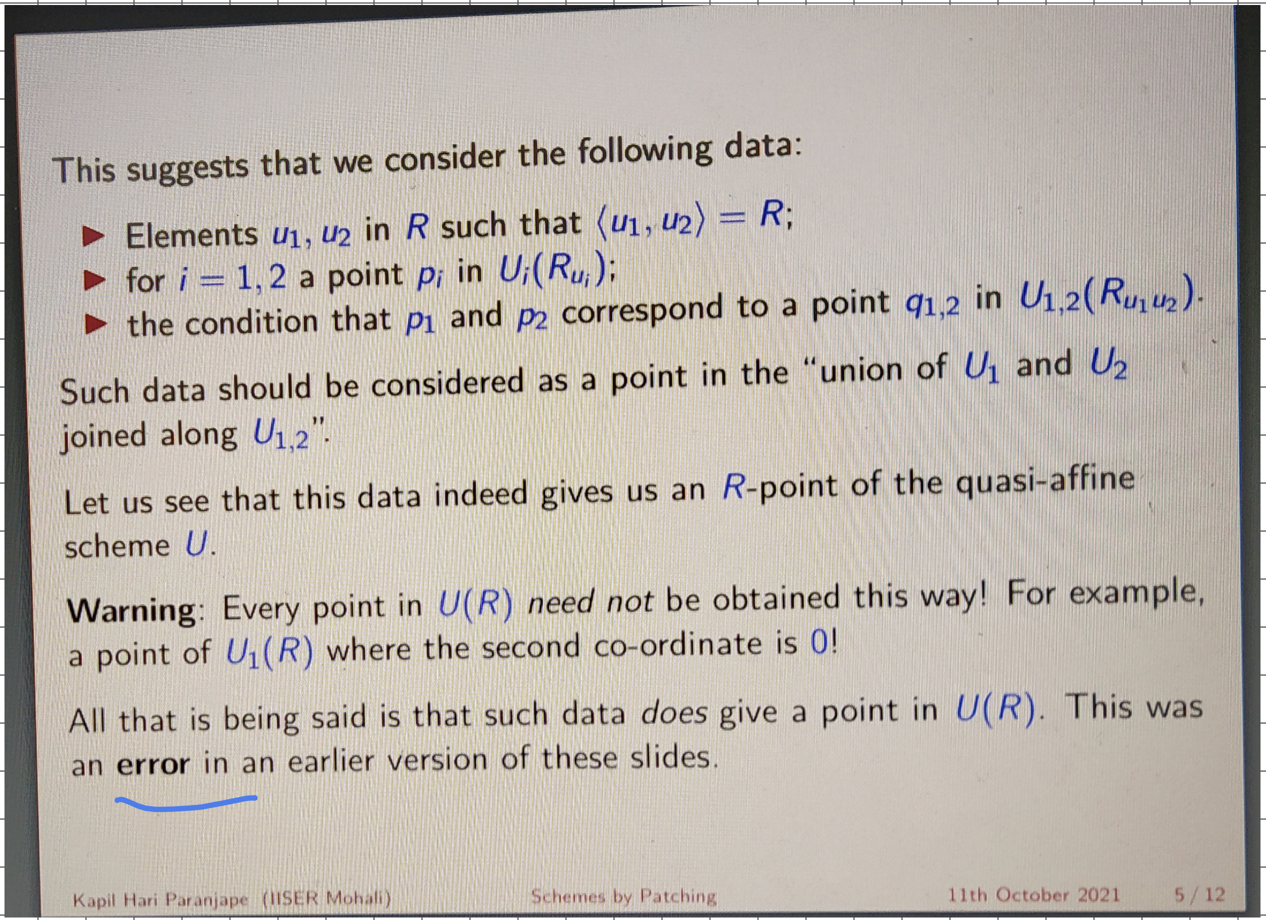
If $\underline{U_1(R_{a_1})} \in \mathbb{F}_2$ $U_2(R_{a_2})$ $\to$ $\underline{(a_1, a_2)} \in U_{1,2}(R_{a_1 a_2})$

$(a_1, a_2, 1/a_1) \in U_1(R_{a_1})$

$(a_1, a_2, 1/a_2) \in U_2(R_{a_2})$

$(a_1, a_2) \in Q(R)$

Prakash



This suggests that we consider the following data:

- Elements u_1, u_2 in R such that $\langle u_1, u_2 \rangle = R$;
- for $i = 1, 2$ a point p_i in $U_i(R_{u_i})$;
- the condition that p_1 and p_2 correspond to a point $q_{1,2}$ in $U_{1,2}(R_{u_1 u_2})$.

Such data should be considered as a point in the "union of U_1 and U_2 joined along $U_{1,2}$".

Let us see that this data indeed gives us an R-point of the quasi-affine scheme U.

Warning: Every point in $U(R)$ *need not* be obtained this way! For example, a point of $U_1(R)$ where the second co-ordinate is 0!

All that is being said is that such data *does* give a point in $U(R)$. This was an **error** in an earlier version of these slides.

Kapil Hari Paranjape (IISER Mohali) Schemes by Patching 11th October 2021 5 / 12

$U_1 = A(x_1, x_2, u_1; u_1x_1 - 1)$

$\mathbb{A}^2 = A(x_1, x_2;)$

$U_1 \longrightarrow \mathbb{A}^2$

$(x_1, x_2, u_1) \longrightarrow (x_1, x_2)$

~~$u_1 = \frac{1}{x_1}$~~

① There is a morphism of functors $U_1 \to \mathbb{A}^2$

② for any R $U_1(R) \to \mathbb{A}^2(R)$ is 1-1

① & ② $\Rightarrow$ $U_1 \to \mathbb{A}^2$ is a subfunctor

$U_1 = \underline{A(x_1, x_2;; x_1)} \hookrightarrow \mathbb{A}^2$

$a_1, a_2 \in R$; $\langle a_1 \rangle_R = R \iff a_1$ is a unit

$u_1 a_1 = 1$

$U_1 = A(x_1, x_2;; x_1)$.

$= \boxed{A(x_1, x_2, u_1; u_1x_1 - 1)}$.

R is a topol. ring

2) in slide 5 , "union of U1 and U2 joined along U1,2" can you explain it. i am trying to understan by example considering u1(Z2) , u2(Z3) and then join them but it not clear to me how to do that.

$p_1 = U_1(R_{n_1})$ ~ $p_2 \in U_2(R_{n_2})$

$\| \qquad\qquad \|$

$(a_1, b_1)\ \cancel{b_1}/_{q_1})$ $\qquad (a_2, b_2)\ \cancel{d_1}/_{b_2})$ $\qquad b_2 d_1 = 1$

$a_1 c_1 = 1$

$U \quad \boxed{R_{n_1,n_2}}$

$(q_1, q_2) \in Q(R) \subset \mathbb{A}^2(R)$

$\mathbb{Z}_1$

$(q_1, q_2) = (a_1, b_1) \in U_1(R_{n_1})$

$\subset \mathbb{A}^2(R_{n_1})$

$\|$ly for R_{n_2}

$U_1 \amalg \exists U_2 \; U_1 \amalg U_2$

$(1, 2) \in U_1(\mathbb{Z})$

$\cap$

$Q(\mathbb{Z})$

$U_2(\mathbb{Z})$

$N \in U(R_{n_1}) \; \Delta_{U_1} \amalg \Delta_{U_2} \amalg U_{1,2}(R) \amalg U_{2,1}$

$\supset \quad \supset \quad \supset \quad \supset$

$U_1 \times U_1 \quad U_2 \times U_2 \quad U_1 \times U_2 \quad U_2 \times U_1$

$\hookrightarrow U \times U$

$\underline{p_1 \in U(R_{n_1})}$; $p_2 \in U(R_{n_2})$

$\underline{(p_1, p_2)} \in U(R_{n_1,n_2}) \times U(R_{n_1,n_2})$

$\hookrightarrow V(R_{n_1,n_2}) > U_{1,2}\underline{(R_{n_1,n_2})}$

$\Rightarrow \quad \underline{q \in Q(R)}$

$\cancel{p_1}$ $\quad p_1 \in U(R_{n_1}) = U_1(R_{n_1}) \amalg U_1(R_{n_2})$

$p_2 \in U(R_{n_2}) = U_1(R_{n_2}) \amalg U_2(R_{n_2})$

$(p_1, p_2) \in U_{1,2}(R_{n_1,n_2})$

$V \hookrightarrow U \times U$ is an equivalence relation

$(U/V)(R)$

$R_{n_1}, R_{n_2} \cdots ; R_{n_k} \quad \langle n_1, \ldots, n_k \rangle = R$

$p_1 \cdot p_2 \cdots p_k \in$

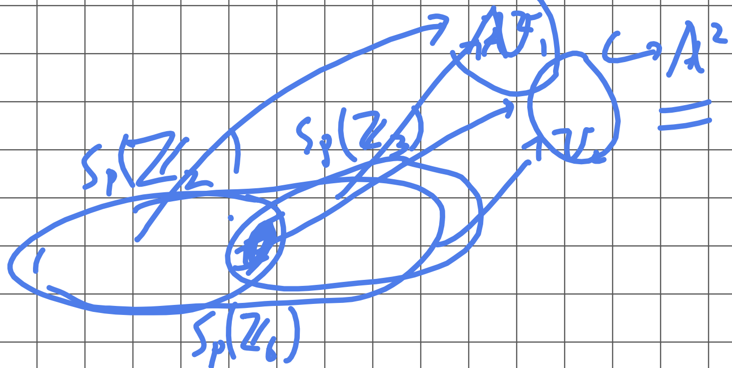
$\in U(R_{n_i})$

$\langle 2, 3 \rangle = \mathbb{Z}$

$(2,3) \in Q(\mathbb{Z})$ $\qquad (2,3) \notin U_1(\mathbb{Z})$

$\notin U_2(\mathbb{Z})$

$\mathbb{Z}_2 \qquad \mathbb{Z}_3.$



$U_1 \xrightarrow{f_1} X$

$U_2 \xrightarrow{f_2} X$

$U_1 \cup U_2 \longrightarrow X$

$f_1|_{U_1 \cap U_2} = f_2|_{U_1 \cap U_2}$

$f_1 : U_1 \cup U_2 \longrightarrow X.$

$X = A(x_1, \dots, x_p; f_1, \dots, f_q; \underline{h_1, \dots, h_s})$

$Y = A(x_1, \dots, x_p; g_1, \dots, g_r); k_1, \dots, k_t)$

$X \cap Y = A(x_1, \dots, x_p; f_1, \dots, f_q, g_1, \dots, g_r)$

$\overline{X} = A(x_1, \dots, x_p; f_1, \dots, f_q)$

$\underline{X} = A(x_1, \dots, x_p; h_1, \dots, h_s)$

$\left.\right\}$ $X =$ "$\overline{X} - \underline{X}$"

$X \cap Y = (\overline{X} \cap \overline{Y}) - (\underline{\underline{X} \cup \underline{Y}})$

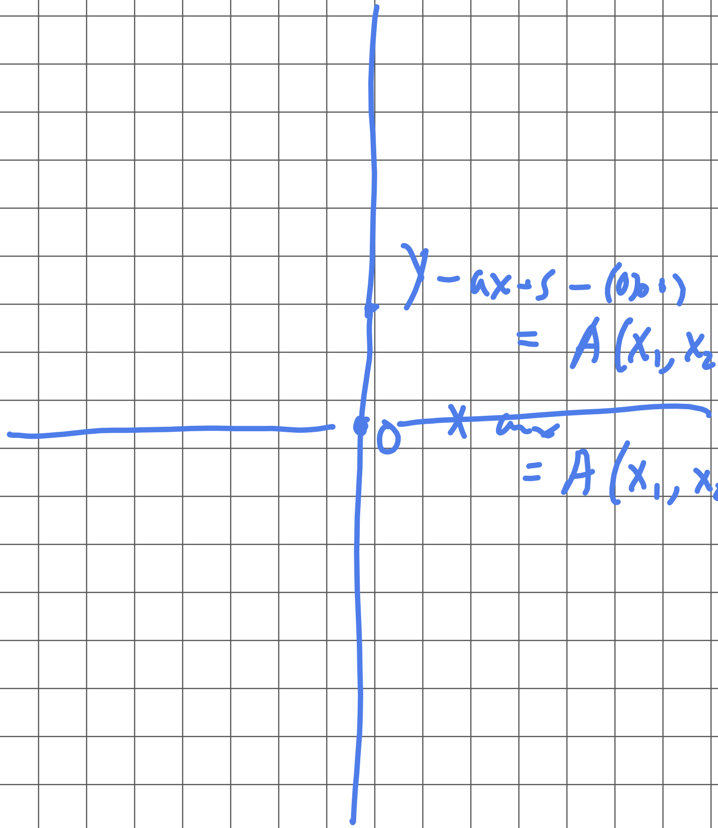
$\underline{a} = (a_1, \dots, a_p)$ $\langle f_i(\underline{a}) \rangle = 0$

$\langle h_i(\underline{a}) \rangle = R.$

$\langle h_i k_j(\underline{a}) \rangle = R$ $\left[\begin{array}{l} \langle g_j(\underline{a}) \rangle = 0 \\ \langle h_i(\underline{a}) \rangle = R \end{array}\right.$

$X \cap Y = A(x_1, \dots, x_p; f_1, \dots, f_q, g_1, \dots, g_r;$
$\underline{h_i k_1, \dots, h_s k_t))}$.

Unión.

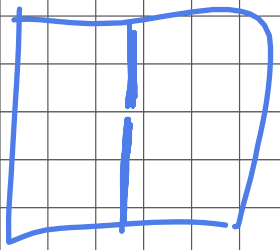


$= A(x_1, x_2; x_2; x_1, x_2)$

$= A(x_1, x_2; x_2; x_1, x_2)$

$\mathbb{A}^2 \setminus \{x\text{-axis}\} = A(x_1, x_2; ; x_2)$

$\{(0,0)\} = A(x_1, x_2; x_1, x_2)$



Objects

6 $\mathbb{N}$. k^i $\boxed{\text{"implementation"}}$

Morphisms.

$i \to j$ ~~$i<j$~~ $i>j$

Different categories

$[0,\dots,i] \xrightarrow[f]{} [0,\dots,j]$

$k^i \to k^j$ $j \times i$ matrices

But objects matter less!

$\{\underset{e}{(1)}, \underset{a}{(12)}, \underset{b}{(34)}, \underset{c}{(12)(34)}\} < S_4$

a	e
b	c
c	b

$Sp(S)$ S

$\underset{\sim}{Aff}' = \underset{\sim}{CRing}^{opp}$

Next two slides from Lecture 16 handwritten extra explanation.

$$\underset{\sim}{\text{Scheme}} \xrightarrow{\sigma} \text{Topspace}$$

$$\underset{\sim}{\text{Aff}} \qquad U \qquad \mathrm{Mor}(U, \mathbb{A}^1) = R \qquad \mathrm{Spec}(R) = \underline{\underline{\sigma(U)}}$$

$$E \rightrightarrows U = \coprod_i U_i \to X \qquad R^{(i)} = \mathrm{Mor}(U_i, \mathbb{A}^1)$$

$$E = \coprod_{i,j} V_{i,j} \qquad V_{i,j} \to U_i \times U_j \qquad \coprod_i \mathrm{Spec}(R^{(i)}) = \sigma(U_i)$$

$Q(R,I) = \mathrm{Spec}(R) \setminus \mathrm{Spec}(R/I)$ open $\subseteq \mathrm{Spec}(R)$

Subspace topology.

$$U' = \coprod_i \mathrm{Spec}(R^{(i)})$$

$$E' = \coprod_{i,j} \sigma(V_{i,j})$$

- $E' \subset U' \times U'$ is an equivalence relation.
- $E' \to U'$ is [illegible]

$\sigma(X) = U'/E'$ as a top. space.

$X \to X_1'$ is a morphism of schemes.

$\sigma(X) \to \sigma(X_1)$ is a homeomorphism

$\not\Rightarrow$ $X \to X'$ is an isomorphism.

$$\mathrm{Sp}(\mathbb{Z}[X,Y]/(X^2)) \leftarrow \mathrm{Sp}(\mathbb{Z}[X,Y]/(X) = \mathbb{Z}[Y])$$

Spec $\longleftrightarrow$ Spec

is not an isom of schemes.

$$\mathrm{Sp}(R)(T) = \underline{\mathrm{Hom}(R,T)}$$

$$\mathrm{Sp}(R) : \underline{\underset{\sim}{\mathrm{Rings}} \to \underset{\sim}{\mathrm{Set}}}$$

~~$\mathrm{Sp}(R)(\mathrm{Hom}(X))$~~ $= R$.

$$\mathrm{Hom}(\mathrm{Sp}(R), \mathbb{A}^1) = R$$

$$\| $$

$$\mathrm{Hom}(\mathbb{Z}[X], R)$$

"Spec(R)" $\underline{\text{top space}}$

Sp(R) – functor sheaf

$\underline{\text{Ringed space}}$

$$\mathrm{Spec}(R) \to \mathrm{Spec}(S)$$

$$\Updownarrow$$

$$\underline{S \to R}$$

$$\mathbb{A}^1 \to \mathbb{A}^2$$

$$t \mapsto (t^2, t^3) = (x, y)$$

morphism $\to$

$$(x^3 - y^2)$$

$$\sigma\big(\underline{\mathbb{A}(x, y; x^3 - y^2)}\big)$$

induced a bijection $\sigma(\mathbb{A}^1)$ hom

$$\frac{\mathbb{C}[x,y]}{(x^3-y^2)} \hookrightarrow \underline{\mathbb{C}[t]}$$

$$x \longrightarrow t^2$$

$$y \longrightarrow t^3$$

$$\operatorname{Spec}(\mathbb{C}[t]) \to \operatorname{Spec}(\mathbb{C}[x,y]/(x^3-y^2))$$

Check this is a homeomorph:

Finite sets are closed sets.