3.4. Carry out a similar exercise to the one above, assuming α'' is an isomorphism.
3.5. Use the *universal property* of the direct sum to show that

$$(A_1 \oplus A_2) \oplus A_3 \cong A_1 \oplus (A_2 \oplus A_3).$$

3.6. Show that $\mathbb{Z}_m \oplus \mathbb{Z}_n = \mathbb{Z}_{mn}$ if and only if m and n are mutually prime.
3.7. Show that the following statements about the exact sequence

$$0 \to A' \xrightarrow{\alpha'} A \xrightarrow{\alpha''} A'' \to 0$$

of Λ-modules are equivalent:
(i) there exists $\mu: A'' \to A$ with $\alpha'' \mu = 1$ on A'';
(ii) there exists $\varepsilon: A \to A'$ with $\varepsilon \alpha' = 1$ on A';
(iii) $0 \to \mathrm{Hom}_\Lambda(B, A') \xrightarrow{\alpha'_*} \mathrm{Hom}_\Lambda(B, A) \xrightarrow{\alpha''_*} \mathrm{Hom}_\Lambda(B, A'') \to 0$ is exact for all B;
(iv) $0 \to \mathrm{Hom}_\Lambda(A'', C) \xrightarrow{\alpha''^*} \mathrm{Hom}_\Lambda(A, C) \xrightarrow{\alpha'^*} \mathrm{Hom}_\Lambda(A', C) \to 0$ is exact for all C;
(v) there exists $\mu: A'' \to A$ such that $\langle \alpha', \mu \rangle : A' \oplus A'' \xrightarrow{\sim} A$.

3.8. Show that if $0 \to A' \xrightarrow{\alpha'} A \xrightarrow{\alpha''} A'' \to 0$ is pure and if A'' is a direct sum of cyclic groups then statement (i) above holds (see Exercise 2.7).

4. Free and Projective Modules

Let A be a Λ-module and let S be a subset of A. We consider the set A_0 of all elements $a \in A$ of the form $a = \sum_{s \in S} \lambda_s s$ where $\lambda_s \in \Lambda$ and $\lambda_s \neq 0$ for only a finite number of elements $s \in S$. It is trivially seen that A_0 is a submodule of A; hence it is the smallest submodule of A containing S.

If for the set S the submodule A_0 is the whole of A, we shall say that S is a set of *generators* of A. If A admits a finite set of generators it is said to be *finitely generated*. A set S of generators of A is called a *basis* of A if every element $a \in A$ may be expressed *uniquely* in the form $a = \sum_{s \in S} \lambda_s s$ with $\lambda_s \in \Lambda$ and $\lambda_s \neq 0$ for only a finite number of elements $s \in S$. It is readily seen that a set S of generators is a basis if and only if it is *linearly independent*, that is, if $\sum_{s \in S} \lambda_s s = 0$ implies $\lambda_s = 0$ for all $s \in S$. The reader should note that not every module possesses a basis.

Definition. If S is a basis of the Λ-module P, then P is called *free on the set S*. We shall call P *free* if it is free on some subset.

Proposition 4.1. *Suppose the Λ-module P is free on the set S. Then $P \cong \bigoplus_{s \in S} \Lambda_s$ where $\Lambda_s = \Lambda$ as a left module for $s \in S$. Conversely, $\bigoplus_{s \in S} \Lambda_s$ is free on the set $\{1_{\Lambda_s}, s \in S\}$.*

Proof. We define $\varphi: P \to \bigoplus_{s \in S} \Lambda_s$ as follows: Every element $a \in P$ is expressed uniquely in the form $a = \sum_{s \in S} \lambda_s s$; set $\varphi(a) = (\lambda_s)_{s \in S}$. Conversely, for $s \in S$ define $\psi_s : \Lambda_s \to P$ by $\psi_s(\lambda_s) = \lambda_s s$. By the universal property of the direct sum the family $\{\psi_s\}$, $s \in S$, gives rise to a map $\psi = \langle \psi_s \rangle : \bigoplus_{s \in S} \Lambda_s \to P$. It is readily seen that φ and ψ are inverse to each other. The remaining assertion immediately follows from the construction of the direct sum. □

The next proposition yields a universal characterization of the free module on the set S.

Proposition 4.2. *Let P be free on the set S. To every Λ-module M and to every function f from S into the set underlying M, there is a unique Λ-module homomorphism $\varphi : P \to M$ extending f.*

Proof. Let $f(s) = m_s$. Set $\varphi(a) = \varphi\left(\sum_{s \in S} \lambda_s s\right) = \sum_{s \in S} \lambda_s m_s$. This obviously is the only homomorphism having the required property. □

Proposition 4.3. *Every Λ-module A is a quotient of a free module P.*

Proof. Let S be a set of generators of A. Let $P = \bigoplus_{s \in S} \Lambda_s$ with $\Lambda_s = \Lambda$ and define $\varphi : P \to A$ to be the extension of the function f given by $f(1_{\Lambda_s}) = s$. Trivially φ is surjective. □

Proposition 4.4. *Let P be a free Λ-module. To every surjective homomorphism $\varepsilon : B \twoheadrightarrow C$ of Λ-modules and to every homomorphism $\gamma : P \to C$ there exists a homomorphism $\beta : P \to B$ such that $\varepsilon\beta = \gamma$.*

Proof. Let P be free on S. Since ε is surjective we can find elements $b_s \in B$, $s \in S$ with $\varepsilon(b_s) = \gamma(s)$, $s \in S$. Define β as the extension of the function $f : S \to B$ given by $f(s) = b_s$, $s \in S$. By the uniqueness part of Proposition 4.2 we conclude that $\varepsilon\beta = \gamma$. □

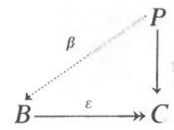
To emphasize the importance of the property proved in Proposition 4.4 we make the following remark: Let $A \overset{\mu}{\rightarrowtail} B \overset{\varepsilon}{\twoheadrightarrow} C$ be a short exact sequence of Λ-modules. If P is a free Λ-module Proposition 4.4 asserts that every homomorphism $\gamma : P \to C$ is induced by a homomorphism $\beta : P \to B$. Hence using Theorem 2.1 we can conclude that the induced sequence

$$0 \to \mathrm{Hom}_\Lambda(P, A) \xrightarrow{\mu_*} \mathrm{Hom}_\Lambda(P, B) \xrightarrow{\varepsilon_*} \mathrm{Hom}_\Lambda(P, C) \to 0 \tag{4.1}$$

is exact, i.e. that ε_* is surjective. Conversely, it is readily seen that exactness of (4.1) for all short exact sequences $A \rightarrowtail B \twoheadrightarrow C$ implies for the module P the property asserted in Proposition 4.4 for P a free module. Therefore there is considerable interest in the class of modules having this property. These are by definition the projective modules:

Definition. A Λ-module P is *projective* if to every surjective homomorphism $\varepsilon : B \twoheadrightarrow C$ of Λ-modules and to every homomorphism $\gamma : P \to C$ there exists a homomorphism $\beta : P \to B$ with $\varepsilon\beta = \gamma$. Equivalently, to any homomorphisms ε, γ with ε surjective in the diagram below there exists

β such that the triangle



is commutative.

As mentioned above, every free module is projective. We shall give some more examples of projective modules at the end of this section.

Proposition 4.5. *A direct sum* $\bigoplus_{i\in I} P_i$ *is projective if and only if each* P_i *is.*

Proof. We prove the proposition only for $A = P \oplus Q$. The proof in the general case is analogous. First assume P and Q projective. Let $\varepsilon: B \twoheadrightarrow C$ be surjective and $\gamma: P \oplus Q \to C$ a homomorphism. Define $\gamma_P = \gamma\, \iota_P: P \to C$ and $\gamma_Q = \gamma\, \iota_Q: Q \to C$. Since P, Q are projective there exist β_P, β_Q such that $\varepsilon\beta_P = \gamma_P$, $\varepsilon\beta_Q = \gamma_Q$. By the universal property of the direct sum there exists $\beta: P \oplus Q \to B$ such that $\beta\,\iota_P = \beta_P$ and $\beta\,\iota_Q = \beta_Q$. It follows that $(\varepsilon\beta)\,\iota_P = \varepsilon\beta_P = \gamma_P = \gamma\,\iota_P$ and $(\varepsilon\beta)\,\iota_Q = \varepsilon\beta_Q = \gamma_Q = \gamma\,\iota_Q$. By the uniqueness part of the universal property we conclude that $\varepsilon\beta = \gamma$. Of course, this could be proved using the explicit *construction* of $P \oplus Q$, but we prefer to emphasize the universal property of the direct sum.

Next assume that $P \oplus Q$ is projective. Let $\varepsilon: B \twoheadrightarrow C$ be a surjection and $\gamma_P: P \to C$ a homomorphism. Choose $\gamma_Q: Q \to C$ to be the zero map. We obtain $\gamma: P \oplus Q \to C$ such that $\gamma\iota_P = \gamma_P$ and $\gamma\iota_Q = \gamma_Q = 0$. Since $P \oplus Q$ is projective there exists $\beta: P \oplus Q \to B$ such that $\varepsilon\beta = \gamma$. Finally we obtain $\varepsilon(\beta\,\iota_P) = \gamma\,\iota_P = \gamma_P$. Hence $\beta\,\iota_P: P \to B$ is the desired homomorphism. Thus P is projective; similarly Q is projective. □

In Theorem 4.7 below we shall give a number of different characterizations of projective modules. As a preparation we define:

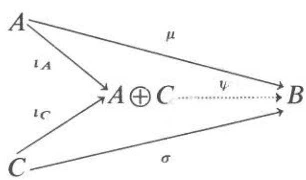
Definition. A short exact sequence $A \overset{\mu}{\rightarrowtail} B \overset{\varepsilon}{\twoheadrightarrow} C$ of Λ-modules *splits* if there exists a left inverse to ε, i.e. a homomorphism $\sigma: C \to B$ such that $\sigma\varepsilon = 1_C$. The map σ is then called a *splitting.*

We remark that the sequence $A \overset{\iota_A}{\rightarrowtail} A \oplus C \overset{\pi_C}{\twoheadrightarrow} C$ is exact, and splits by the homomorphism ι_C. The following lemma shows that all split short exact sequences of modules are of this form (see Exercise 3.7).

Lemma 4.6. *Suppose that* $\sigma: C \to B$ *is a splitting for the short exact sequence* $A \overset{\mu}{\rightarrowtail} B \overset{\varepsilon}{\twoheadrightarrow} C$. *Then* B *is isomorphic to the direct sum* $A \oplus C$. *Under this isomorphism,* μ *corresponds to* ι_A *and* σ *to* ι_C.

In this case we shall say that C (like A) is a *direct summand* in B.

Proof. By the universal property of the direct sum we define a map ψ as follows



Then the diagram

$$\begin{array}{ccccc} A & \xrightarrow{\iota_A} & A \oplus C & \xrightarrow{\pi_C} & C \\ \| & & \downarrow{\scriptstyle \psi} & & \| \\ A & \xrightarrow{\mu} & B & \xrightarrow{\varepsilon} & C \end{array}$$

is commutative; the left hand square trivially is; the right hand square is by $\varepsilon\psi(a, c) = \varepsilon(\mu a + \sigma c) = 0 + \varepsilon\sigma e = c$, and $\pi_C(a, c) = c$, $a \in A$, $c \in C$. By Lemma 1.1 ψ is an isomorphism. □

Theorem 4.7. *For a Λ-module P the following statements are equivalent:*

(1) *P is projective;*

(2) *for every short exact sequence $A \overset{\mu}{\rightarrowtail} B \overset{\varepsilon}{\twoheadrightarrow} C$ of Λ-modules the induced sequence*

$$0 \rightarrow \mathrm{Hom}_\Lambda(P, A) \xrightarrow{\mu_*} \mathrm{Hom}_\Lambda(P, B) \xrightarrow{\varepsilon_*} \mathrm{Hom}_\Lambda(P, C) \rightarrow 0$$

is exact;

(3) *if $\varepsilon: B \twoheadrightarrow P$ is surjective, then there exists a homomorphism $\beta: P \rightarrow B$ such that $\varepsilon\beta = 1_P$;*

(4) *P is a direct summand in every module of which it is a quotient;*

(5) *P is a direct summand in a free module.*

Proof. (1)⇒(2). By Theorem 2.1 we only have to show exactness at $\mathrm{Hom}_\Lambda(P, C)$, i.e. that ε_* is surjective. But since $\varepsilon: B \rightarrow C$ is surjective this is asserted by the fact that P is projective.

(2)⇒(3). Choose as exact sequence $\ker\varepsilon \rightarrowtail B \overset{\varepsilon}{\twoheadrightarrow} P$. The induced sequence

$$0 \rightarrow \mathrm{Hom}_\Lambda(P, \ker\varepsilon) \rightarrow \mathrm{Hom}_\Lambda(P, B) \xrightarrow{\varepsilon_*} \mathrm{Hom}_\Lambda(P, P) \rightarrow 0$$

is exact. Therefore there exists $\beta: P \rightarrow B$ such that $\varepsilon\beta = 1_P$.

(3)⇒(4). Let $P \cong B/A$, then we have an exact sequence $A \rightarrowtail B \overset{\varepsilon}{\twoheadrightarrow} P$. By (3) there exists $\beta: P \rightarrow B$ such that $\varepsilon\beta = 1_P$. By Lemma 4.6 we conclude that P is a direct summand in B.

(4)⇒(5). By Proposition 4.3 P is a quotient of a free module P'. By (4) P is a direct summand in P'.

(5)⇒(1). By (5) $P' \cong P \oplus Q$, where P' is a free module. Since free modules are projective, it follows from Proposition 4.5 that P is projective. □

Next we list some examples:

(a) If $\Lambda = K$, a field, then every K-module is free, hence projective.

(b) By Exercise 2.2 and (2) of Theorem 4.7, $\mathbb{Z}_n$ is not projective as a module over the integers. Hence a finitely generated abelian group is projective if and only if it is free.

(c) Let $\Lambda = \mathbb{Z}_6$, the ring of integers modulo 6. Since $\mathbb{Z}_6 = \mathbb{Z}_3 \oplus \mathbb{Z}_2$ as a $\mathbb{Z}_6$-module, Proposition 4.5 shows that $\mathbb{Z}_2$ as well as $\mathbb{Z}_3$ are projective $\mathbb{Z}_6$-modules. However, they are plainly not free $\mathbb{Z}_6$-modules.

Exercises:

4.1. Let V be a vector space of countable dimension over the field K. Let $\Lambda = \mathrm{Hom}_K(V, V)$. Show that, as K-vector spaces V, is isomorphic to $V \oplus V$. We therefore obtain

$$\Lambda = \mathrm{Hom}_K(V, V) \cong \mathrm{Hom}_K(V \oplus V, V) \cong \mathrm{Hom}_K(V, V) \oplus \mathrm{Hom}_K(V, V) = \Lambda \oplus \Lambda .$$

Conclude that, in general, the free module on a set of n elements may be isomorphic to the free module on a set of m elements, with $n \neq m$.

4.2. Given two projective Λ-modules P, Q, show that there exists a *free* Λ-module R such that $P \oplus R \cong Q \oplus R$ is free. (Hint: Let $P \oplus P'$ and $Q \oplus Q'$ be free. Define $R = P' \oplus (Q \oplus Q') \oplus (P \oplus P') \oplus \cdots \cong Q' \oplus (P \oplus P') \oplus (Q \oplus Q') \oplus \cdots$.)

4.3. Show that $\mathbb{Q}$ is not a free $\mathbb{Z}$-module.

4.4. Need a direct product of projective modules be projective?

4.5. Show that if $0 \to N \to P \to A \to 0$, $0 \to M \to Q \to A \to 0$ are exact with P, Q projective, then $P \oplus M \cong Q \oplus N$. (Hint: Use Exercise 3.4.)

4.6. We say that A has a *finite presentation* if there is a short exact sequence $0 \to N \to P \to A \to 0$ with P finitely-generated projective and N finitely-generated. Show that

(i) if A has a finite presentation, then, for every exact sequence

$$0 \to R \to S \to A \to 0$$

with S finitely-generated, R is also finitely-generated;

(ii) if A has a finite presentation, it has a finite presentation with P free;

(iii) if A has a finite presentation every presentation $0 \to N \to P \to A \to 0$ with P projective, N finitely-generated is finite, and every presentation $0 \to N \to P \to A \to 0$ with P finitely-generated projective is finite:

(iv) if A has a presentation $0 \to N_1 \to P_1 \to A \to 0$ with P_1 finitely-generated projective, and a presentation $0 \to N_2 \to P_2 \to A \to 0$ with P_2 projective, N_2 finitely-generated, then A has a finite presentation (indeed, both the given presentations are finite).

4.7. Let $\Lambda = K(x_1, \ldots, x_n, \ldots)$ be the polynomial ring in countably many indeterminates $x_1, \ldots, x_n, \ldots$ over the field K. Show that the ideal I generated by $x_1, \ldots, x_n, \ldots$ is not finitely generated. Hence we may have a presentation $0 \to N \to P \to A \to 0$ with P finitely generated projective and N *not* finitely-generated.

5. Projective Modules over a Principal Ideal Domain

Here we shall prove a rather difficult theorem about principal ideal domains. We remark that a very simple proof is available if one is content to consider only finitely generated Λ-modules; then the theorem forms a part of the fundamental classical theorem on the structure of finitely generated modules over principal ideal domains.

Recall that a principal ideal domain Λ is a commutative ring without divisors of zero in which every ideal is principal, i.e. generated by