Ayush

when, we are trying to construct disjoint union of Functors F and G from C.R. to set and we defined it $(F \sqcup G)(R) = F(R) \sqcup G(R)$

In case of $\mathbb{Z}$ affine scheme X and Y as Functors and $R = \{0\}$ we have shown some contradiction

my questions are

(i) How $R = \{0\}$ represent Empty space

(ii) we have use of word Geometric Functors F and G. so, what do we mean by geometric Functor. Intutively, I am thinking about $\mathbb{Z}$ affine scheme. please elaborate it.

1. The empty space ϕ has the property that there is a unique map $\phi \to X$ for any space X.

$\updownarrow$

we note that there is a unique map $R \to \{0\}$ for any ring R.

2. There is only one function on the empty space! $(0 = 1 = \ldots\ldots)$

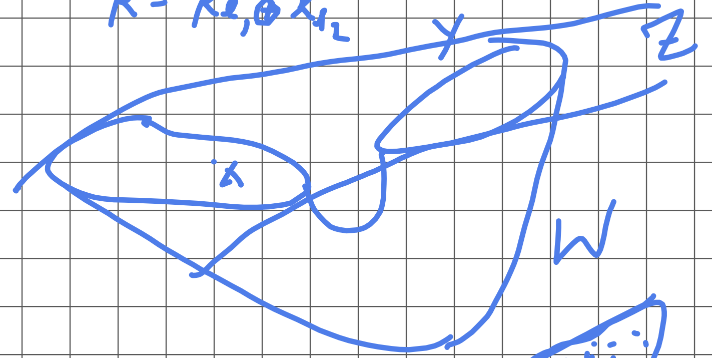
3. Geometric Functor is a loose term for a functor which arise in a Geometric problem. E.g. $\mathbb{Z}$-affine scheme, $\mathbb{A}^n$, $\mathbb{P}^n$ etc.

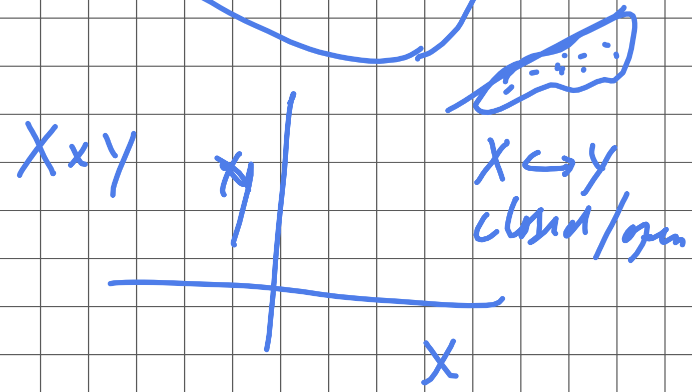
$X(R) \sqcup Y(R)$

$\phi \to (X \sqcup Y)$ [unique]

$(X \sqcup Y)(R) \neq X(R) \sqcup Y(R)$ | $R = \{0\}$

$R = R_{e_1} \oplus R_{e_2}$





The scheme $X = \mathbb{A}(x,y,z;\ xy - z(z-1))$
is a "quadric surface" roughly like a "hyperboloid"

$\langle x, z\rangle = I$ gives $Y = \mathbb{A}(x,y,z;\ x,z] \subset X$
$\cong \mathbb{A}^1$

The picture is that cutting a hyperboloid along a line contained in it allows me to "unfold" it into a plane

in fact let $Z = \mathbb{A}(x,t;\,) = \mathbb{A}^2$

$Z \longrightarrow X$ given by

$(x,t) \longmapsto (x, (1+xt)t, 1+xt) = (x,y,z)$

$z(z-1) = (1+xt)((1+xt)-1) = (1+xt)xt = xy$

Claim:- The "complement" of Y in X is Z

$$\mathcal{O}(X) = \frac{\mathbb{Z}[x,y,z]}{\langle xy - z(z-1)\rangle} \longrightarrow \mathcal{O}(Z) = \mathbb{Z}[x,t]$$

$$(x,y,z) \longmapsto (x, (1+xt)t, 1+xt)$$

$\mathcal{O}(X) \xrightarrow{f} R$

s.t. $\langle f(x), f(z)\rangle_R = R$

$\Rightarrow$ $a, b \in R$ s.t. $af(x) + bf(z) = 1$

$\Rightarrow$ $af(x)f(y) + bf(z)f(y) = f(y)$

$af(z)f(z-1) + bf(z)f(y) = f(y)$ $\Longrightarrow$

This suggest $\tilde{f}(t) = af(z-1) + bf(y)$
$= a(f(z)-1) + bf(y)$

Now, we check that this gives

$\mathcal{O}(Z) = \mathbb{Z}[x,t] \longrightarrow R$ $\quad x \mapsto f(x),\ t \mapsto \tilde{f}(t)$

s.t. $\underset{\mathcal{O}(X)}{\mathbb{Z}[x,y,z]} \to \underset{\mathcal{O}(Z)}{\mathbb{Z}[x,t]} \to R$ is given map f
"$X - Y$"

Note that $Z \subset X$ is open & affine
but there is no $\alpha \in \mathcal{O}(X)$ s.t.
$\boxed{Z = X_\alpha}$ $\left(\mathcal{O}(Z) = \dfrac{\mathcal{O}(X)[u]}{(\alpha u - 1)}\right)$

$p \in \mathbb{P}^1(\mathcal{O}(X))$

$p \neq (a : b)$ $\quad a, b \in \mathcal{O}(X)$

$\boxed{R,\ \text{local}}$

Prakash

① Let X and Y be affine scheme.

then Morphism $f: X \longrightarrow Y$

means for any Ring R;

$$f(R) : X(R) \longrightarrow Y(R)$$

{ set theoretic map which maps zeros of X to zeros of Y }

Given $O(Y) \xrightarrow{g} O(X')$

and ~~$f(R): X(R) \to Y(R)$~~

to show $\exists\ f : X \longrightarrow Y$

i.e to show for any R $f(R) : X(R) \longrightarrow Y(R)$

for any $h \in X(R) = \mathrm{Hom}(O(X), R)$

$\Rightarrow$ so $O(Y) \xrightarrow{g} O(X) \xrightarrow{h} R$

$\Rightarrow \therefore\ h \circ g : O(Y) \longrightarrow R$

$\Rightarrow f(h) = h \circ g \in Y(R)$ Done

$\boxed{\text{but how to go in reverse Direction ??}}$

Given $f : X \to Y$

how to show $\exists\ g : O(Y) \longrightarrow O(X)$

I $X_\cdot \to Y_\cdot$ means $X(R) \to Y(R)$ functorially (comm. diagram)

take $R = \mathcal{O}(X)$ $\quad X(\mathcal{O}(X)) = \mathrm{Hom}(\mathcal{O}(X), \mathcal{O}(X)) \ni id_X$

image of id_X gives $f \in Y(\mathcal{O}(X)) = \mathrm{Hom}(\mathcal{O}(Y), \mathcal{O}(X))$
$= \mathrm{Mor}(X, Y)$

II Given $f : X \to Y$ then $f \in \mathrm{Mor}(X,Y) = \mathrm{Hom}(\mathcal{O}(Y), \mathcal{O}(X))$

So given $g \in X(R) = \mathrm{Hom}(\mathcal{O}(X), R)$

composition gives $\mathcal{O}(Y) \xrightarrow{f} \mathcal{O}(X) \xrightarrow{g} R$

$\therefore\ g \circ f \in Y(R)$

(Yoneda Lemma)

Wikipedia.

The question is: Given X, Y affine schemes.

we have associated $X_\cdot, Y_\cdot$ functors CRing to Set

Given a morphism $X \to Y$ how do we get $X_\cdot \to Y_\cdot$ & vice versa?

CRing ⇝ Set

Set-theoretic constructions
{ |Adapt.| }
"Geometric flavour"

$Sp(\mathbb{Z}[x,y]) = A(x,y;\ ) = A^2$

$A^2(R) = R^2$

$B(R)$ = "algebraic"

"quasi-affine"

$\langle f_1, \dots, f_q \rangle = 0$ identities hold
$\neq$ hold $\langle g_1, \dots, g_r \rangle = R.$

Quasi-affine schemes.

Patching gives more "schemes".

$\mathbb{P}^n$; $\mathbb{P}^\infty$. ...