# General Schemes

## MTH437 — Introduction to Schemes


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# Recall

In the previous two lectures we introduced the notion of a $\mathbb{Z}$-scheme $F$ of finite type and gave some examples.

- $F$ is a sheaf functor from **CRing** to **Set** for which there is sheaf-theoretic surjection $\eta : U \to F$ where $U = A(\mathbf{x}; \mathbf{f})$ is a $\mathbb{Z}$-affine scheme of finite type.
- The kernel $E = \ker \eta$ is a $\mathbb{Z}$-*quasi* affine scheme of finite type. Note that we have a morphisms $E \to U \times U$.
- There is a decomposition $U = \sqcup_{i=1}^{n} U_i$ into affine schemes $U_i$. We then define $V_{i,j}$ as $E \cap U_i \times U_j$ for all $i$ and $j$.
- For each $i$ and $j$ the morphism $V_{i,j} \to U_i$ is a (quasi-affine) open subscheme. (By reflexivity, so is $V_{i,j} \to U_j$.)

We now generalise this a bit further.

# General Affine Schemes

Given a commutative ring $T$, we have introduced a functor $T^{\cdot}$ from **CRing** to **Set**.

For a ring $R$, recall that $T^{\cdot}(R) = \mathrm{Hom}(T, R)$

We have checked that $T^{\cdot}$ is a sheaf functor.

Note that this does not require $T$ to be a finitely presented ring.

Since the "dot" is hard to see, let us use $\mathsf{Sp}(T)$ to denote this functor. We call this an *affine* scheme.

Recall that a morphism $\mathsf{Sp}(T) \to \mathsf{Sp}(U)$ is given by a homomorphism of rings $U \to T$.

# Closed subscheme

Given an ideal $I$ in $T$, we have a homomorphism $T \to T/I$ which gives a morphism $\mathrm{Sp}(T/I) \to \mathrm{Sp}(T)$ of functors.

For a commutative ring $R$, a homomorphism $T/I \to R$ is *uniquely* determined by a homomorphism $T \to R$ such that $I$ goes to $0$ in $R$.

This shows that $\mathrm{Sp}(T/I) \to \mathrm{Sp}(T)$ is a subfunctor.

We say that $\mathrm{Sp}(T/I)$ is an *closed* subscheme of $\mathrm{Sp}(T)$ under this morphism.

Given ideals $I$ and $J$ in $T$, their sum $I + J$ is also an ideal in $T$.

Given a ring $R$, a homomorphism $T/(I + J) \to R$ is *uniquely* determined by a homomorphism $T \to R$ for which the images of $I$ *and* $J$ are $0$.

It follows that $\mathrm{Sp}(T/(I + J))$ is the intersection $\mathrm{Sp}(T/I) \cap \mathrm{Sp}(T/J)$ in $\mathrm{Sp}(T)$.

# Affine open subschemes

Given an element $g$ in a commutative ring $T$, we have a ring homomorphism $T \to T_g$ which gives a morphism $\mathrm{Sp}(T_g) \to \mathrm{Sp}(T)$ of functors.

For a commutative ring $R$, a homomorphism $T_g \to R$ is *uniquely* determined by a homomorphism $T \to R$ such that $g$ goes to a unit in $R$.

This shows that $\mathrm{Sp}(T_g) \to \mathrm{Sp}(T)$ is a subfunctor.

We say that $\mathrm{Sp}(T_g)$ is an affine *open* subscheme of $\mathrm{Sp}(T)$ under this morphism.

If $g$ and $h$ are elements of $T$, then we have $\mathrm{Sp}(T_{gh})$ as an affine open subscheme of $\mathrm{Sp}(T_g)$ and $\mathrm{Sp}(T_h)$.

Given a ring $R$, a homomorphism $T_{gh} \to R$ is *uniquely* determined by a homomorphism $T \to R$ for which the images of $g$ *and* $h$ are units.

It follows that $\mathrm{Sp}(T_{gh})$ is the intersection $\mathrm{Sp}(T_g) \cap \mathrm{Sp}(T_h)$ in $\mathrm{Sp}(T)$.

# Open Subschemes

We have the more general notion of (quasi-affine) open subscheme of an affine scheme.

This is a subfunctor, but is *not*, in general, an affine scheme.

Given an ideal $I$ in the commutative ring $T$, we define the "complement" $D_I$ as the subfunctor

$$D_I(R) = \{f : T \to R : f(I)R = R\}$$

This consists of homomorphisms where the image of $I$ generates the unit ideal.

We regard $D_I \to T$ as an open subfunctor. We call it a quasi-affine scheme.

When $I = \langle g \rangle$ we see that $D_I = \mathrm{Sp}(T_g)$.

# Affine Open Cover

Let us understand the sheaf condition in terms of affine schemes.

Given a ring $T$ and elements $u_1, \ldots, u_k$ which generate the unit ideal.

We have affine open subschemes $\mathrm{Sp}(T_{u_i})$ of $\mathrm{Sp}(T)$.

Moreover, the morphism $\sqcup_i \mathrm{Sp}(T_{u_i}) \to \mathrm{Sp}(T)$ is a *sheaf-theoretic* surjection.

In other words, we can think of this as an *affine open cover* of $\mathrm{Sp}(T)$.

# Sheaf Condition re-done

Suppose that $\sqcup_i \mathrm{Sp}(T_{u_i}) \to \mathrm{Sp}(T)$ is an affine open cover (as above).

As seen above $\mathrm{Sp}(T_{u_i u_i})$ is the intersection $\mathrm{Sp}(T_{u_i}) \cap \mathrm{Sp}(T_{u_j})$ in $\mathrm{Sp}(T)$.

Given a functor $F$ from **CRing** to **Set**, the sheaf condition for $F(T)$ can be written as follows.

**Sheaf Condition in terms of affine open cover**: Given $f_i : \mathrm{Sp}(T_{u_i}) \to F$ for $i = 1, \ldots, k$ such that $f_i$ and $f_j$ agree on the intersection $\mathrm{Sp}(T_{u_i u_j})$, there is a *unique* $f : \mathrm{Sp}(T) \to F$ that restricts to $f_i$ on $\mathrm{Sp}(T_{u_i})$.

This looks very similar to the sheaf condition for continuous maps from a topological space $T$ to a topological space $F$.

# A General Scheme

Informally, a *scheme* is a sheaf functor $F$ which is a quotient of $\sqcup_i \mathrm{Sp}(T^{(i)})$ (a collection of affine schemes) by an equivalence relation of the form $\sqcup_{i,j} V^{(i,j)}$ (a collection of quasi-affine schemes) where each $V^{(i,j)}$ is an open subscheme of $\mathrm{Sp}(R^{(i)})$.

Note that the collections need not be *finite*.

More precisely, we are given:

- An indexed collection $\{U_i = \mathrm{Sp}(R^{(i)})\}_{i \in I}$ of affine schemes. This gives $U = \sqcup_{i \in I} U_i$.
- An equivalence relation $E \subset U \times U$. This gives us $V_{i,j}$ as the intersection $E \cap U_i \times U_j$.
- The projection $\pi_1 : V^{(i,j)} \to U_i$ identifies this as a quasi-affine open subscheme. (It follows that the same is true for $\pi_2 : V^{(i,j)} \to \mathrm{Sp}(R^{(j)})$.)

$F$ is the quotient $U/E$ and we say $F$ is the scheme obtained by patching $U_i$ along $V_{i,j}$.

# A pathological example

We take $U = U_1 \sqcup U_2$ where $U_i \cong \mathbb{A}^n$ for $i = 1, 2$.

We define $V_{1,1}$ and $V_{2,2}$ to be the diagonal inclusions.

We define $V_{1,2}$ to be the inclusion on $\mathbb{A}^n \setminus \{(0, \ldots, 0)\}$ in *both* factors. Similarly, $V_{2,1}$.

It follows that this gives us an equivalence relation on $U$.

The quotient sheaf $F$ consists of two copies of $\mathbb{A}^n$ joined along the complement of the origin. A strange space to study!

# Separated schemes

In order to avoid such pathologies, we can impose the requirement that $V_{i,j}$ is a *closed* subscheme of $U_i \times U_j$ for each $i$ and $j$.

In our construction of quasi-projective schemes we can see that this condition is satisfied.

Note that, this *implies* that $V_{i,j}$ is affine. So $V_{i,j} = \mathrm{Sp}(S^{(i,j)})$ for some commutative ring $S^{(i,j)}$.

So a *separated* scheme is of the form:

- An indexed collection $\{U_i = \mathrm{Sp}(R^{(i)})\}_{i \in I}$ of affine schemes.
- An indexed collection $\{V_{i,j} = \mathrm{Sp}(S^{(i,j)})\}_{(i,j) \in I \times I}$ of closed subschemes contained in $U_i \times U_j$
- This gives $U = \sqcup_{i \in I} U_i$, and $E = \sqcup_{(i,j) \in I \times I} V_{i,j}$, and an inclusion $E \to U \times U$.
- The combined morphism $E$ gives an equivalence relation on the functor $U$.
- $\pi_1 : V_{i,j} \to U_i$ identifies this as an open subscheme.