Projective Schemes

MTH437 — Introduction to Schemes

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Recall

In the previous two lecture we introduced the notion of a $\mathbb{Z}$-scheme F (of finite type) via patching or as a quotient.

- F is a sheaf functor from **CRing** to **Set** for which there is sheaf-theoretic surjection $\eta : U \to F$ where $U = A(\mathbf{x}; \mathbf{f})$ is a $\mathbb{Z}$-affine scheme of finite type.
- The kernel $E = \ker \eta$ is also $\mathbb{Z}$-affine scheme of finite type. Note that we have a morphisms $E \to U \times U$.
- There are decompositions $U = \sqcup_{i=1}^{n} U_i$ and $E = \sqcup_{i,j=1}^{n} V_{i,j}$ into affine schemes so that $E \to U \times U$ is restricts to $V_{i,j} \to U_i \times U_j$ for all i and j.
- For each i and j the morphism $V_{i,j} \to U_i$ is an affine open subscheme. (By reflexivity, so is $V_{i,j} \to U_j$.)

Warning: The earlier version of these slides and the lecture was in **error**.

Note that F is the *sheaf quotient* U/E by the equivalence relation $E \rightrightarrows U$.

Hence, we can also give the alternative description of schemes as follows.

- We are given $\mathbb{Z}$-affine schemes $U = \sqcup_{i=1}^{n} U_i$ and $E = \sqcup_{i,j=1}^{n} V_{i,j}$.
- There is a morphism $E \to U \times U$ that makes $E(R)$ an equivalence relation on $U(R)$ for each ring R. This morphism restricts to morphisms $V_{i,j} \to U_i \times U_j$ for each i and j.
- For each i and j the morphisms $V_{i,j} \to U_i$ is an affine open subscheme. (By reflexivity, so is $V_{i,j} \to U_j$.)

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In this case F is *defined* as the sheaf quotient U/E.

We now give some important examples of schemes.

Projective space

Recall the notion of projective space $\mathbb{P}^n(k)$ over a field k.

It is defined as the quotient of $k^{n+1} \setminus \{(0, \ldots, 0)\}$ by the equivalence relation

$$(a_0, \ldots, a_n) \sim (ua_0, \ldots, ua_n) \text{ where } u \neq 0$$

In order to generalise this definition to arbitrary rings, we need to resolve the following issues:

- $\mathbb{A}^{n+1} \setminus \{(0, \ldots, 0)\}$ is a *quasi*-affine scheme and not an affine scheme for $n > 0$.
- The equivalence classes are *not* discrete/finite.

The first problem can be resolved (partially) by writing $\mathbb{A}^{n+1} \setminus \{(0,\ldots,0)\}$ as the sheaf-theoretic union of $X_i = A(x_0,\ldots,x_n,y_i;x_iy_i - 1)$ which is the open subscheme of $\mathbb{A}^{n+1}$ where x_i is a unit.

In other words, we have a surjection of sheaf functors

$$\sqcup_{i=0}^{n} X_i \to \mathbb{A}^{n+1} \setminus \{(0,\ldots,0)\}$$

Note that the equivalence relation associated to *this surjection* is the disjoint union of $X_{i,j} = A(\mathbf{x}, y_i, y_j; x_iy_i - 1, x_jy_j - 1)$ over all i and j.

However, we *also* need to quotient by the equivalence relation associated with

$$(a_0,\ldots,a_n) \sim (ua_0,\ldots,ua_n) \text{ where } u \text{ is a unit}$$

Note that in X_i, the function x_i is a unit.

Thus, $U_i = A(\mathbf{x}; x_i - 1)$ is a closed subscheme of X_i that “slices” each equivalence class. Note that $U_i \cong \mathbb{A}^n$.

Since we do not want to slice twice, we take

$$U_{i,j} = A(x_0, \ldots, x_n, y_j; x_i - 1, x_j y_j - 1)$$

This is the affine open subscheme of U_i where x_j is a unit.

The morphism $U_{i,j} \to U_j$ is given by

$$(x_0, \ldots, x_n, y_j) \mapsto (y_j x_0, \ldots, y_j x_n)$$

Note that $y_j x_j = 1$ on the right-hand side.

This morphism identifies $U_{i,j}$ with the affine open subset of U_j where $x_i = y_j$ is a unit.

In other words, $U_{i,j}(R)$ as a *subset* of $U_i(R) \times U_j(R)$ consists of pairs of tuples $((a_0, \ldots, a_n), (b_0, \ldots, b_n))$ such that:

- $a_i = 1$ and $b_j = 1$
- a_j and b_i are units such that $a_j b_i = 1$
- $b_i a_k = b_k$ and (equivalently) $a_j b_k = a_k$ for $k = 0, \ldots, n$

It follows that $E = \sqcup_{i,j=1}^{n} U_{i,j}$ gives an equivalence relation on $U = \sqcup_{i=0}^{n} U_i$.

The scheme $\mathbb{P}^n$ is defined as the quotient U/E.

Closed subscheme of $\mathbb{P}^n$

We now introduce a closed subscheme of $\mathbb{P}^n$ as the locus where a collection $f_1, \ldots, f_m$ of *homogeneous* polynomials vanish.

This subscheme will be denoted as $P(\mathbf{x}; \mathbf{f}) = P(x_0, \ldots, x_n; f_1, \ldots, f_m)$.

We can define

$$V_i \cap P(\mathbf{x}; \mathbf{f}) = V_i = A(x_0, \ldots, x_i, \ldots, x_n; x_i - 1, f_1, \ldots, f_m)$$

This is an affine scheme.

Recall that $U_{i,j}(R)$ as a *subset* of $U_i(R) \times U_j(R)$ consists of pairs of tuples $((a_0, \ldots, a_n), (b_0, \ldots, b_n))$ such that:

- $a_i = 1$ and $b_j = 1$
- a_j and b_i are units such that $a_j b_i = 1$
- $b_i a_k = b_k$ and (equivalently) $a_j b_k = a_k$ for $k = 0, \ldots, n$

We calculate:

$$
\begin{aligned}
& f_r(a_0, \ldots, a_n) \\
& = f_r(a_j b_0, \ldots, a_j b_n) && \text{(since } a_k = a_j b_k\text{)} \\
& = a_j^{d_r} f_r(b_0, \ldots, b_n) && \text{(since } f_r \text{ is homogeneous)}
\end{aligned}
$$

where d_r is the degree of f_r.

Since a_j is a unit in R, we see that $f_r(\mathbf{a}) = 0$ if and only if $f_r(\mathbf{b}) = 0$.

This shows that the following intersections are equal inside $U_i(R) \times U_j(R)$

$$
\begin{aligned}
& U_{i,j}(R) \cap (V_i(R) \times V_j(R)) \\
& = U_{i,j}(R) \cap (U_i(R) \times V_j(R)) \\
& = U_{i,j}(R) \cap (V_i(R) \times U_j(R))
\end{aligned}
$$

We call these intersections $V_{i,j}$.

Now define $V = \sqcup_{i=1}^{n} V_i$ and $E_V = \sqcup_{i,j=1}^{n} V_{i,j}$ with the natural pair of morphisms $E_V \rightrightarrows V$.

The above discussion shows that $V_{i,j} \to V_i$ is the affine open set where x_j is a unit.

Thus, the conditions for $E_V \rightrightarrows V$ to have scheme as its sheaf quotient are satisfied.

The projective scheme $P(\mathbf{x}; \mathbf{f})$ is the sheaf quotient V/E_V.

Quasi-projective schemes

This can be further extended to the notion of a quasi-projective scheme

$$P(x_0, \ldots, x_n; f_1, \ldots, f_m; g_1, \ldots, g_\ell)$$

where $f_1, \ldots, f_m, g_1, \ldots, g_\ell$ are homogeneous polynomials.

This is the locus inside $P(x_0, \ldots, x_n; f_1, \ldots, f_m)$ where $g_1, \ldots, g_\ell$ generate the unit ideal.

We define $W_{s,i} = A(x_0, \ldots, x_n, v_s; x_i - 1, f_1, \ldots, f_m, v_s g_s - 1)$.

Note that for a point $(\mathbf{a}, \mathbf{b})$ in $U_{i,j}(R)$, the value of $g_s(\mathbf{a})$ is a unit multiple of $g_s(\mathbf{b})$ for all $s = 1, \ldots, \ell$.

In particular, the condition that g_s is a unit is independent of whether we are looking at the point in U_i or in U_j.

We now take $W_{(s,i),(r,j)}(R)$ to be $U_{i,j}(R) \cap W_{s,i}(R) \times W_{r,j}(R)$ inside $U_i(R) \times U_j(R)$.

From the above discussion, we see that $E_W = \sqcup_{s,i,r,j} W_{(s,i),(r,j)}$ is a $\mathbb{Z}$-affine scheme which is an affine open subscheme of $W = \sqcup_{s,i} W_{s,i}$ under either projection.

Moreover, one checks as above that $E_W \rightrightarrows W$ is an equivalence relation.

The quasi-projective scheme $P(x_0, \ldots, x_n; f_1, \ldots, f_m; g_1, \ldots, g_\ell)$ is the sheaf quotient W/E_W.