Kernel, image and exactness

MTH437 — Introduction to Schemes

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Recall

In the previous lecture we showed how $\mathbb{Z}$-affine schemes can be patched to give a sheaf functor.

Strictly speaking, what we have been calling a $\mathbb{Z}$-affine scheme is actually a $\mathbb{Z}$-*affine scheme of finite type*.

A $\mathbb{Z}$-*scheme of finite type* is a sheaf functor obtained by patching from $\mathbb{Z}$-affine schemes of finite type as described in the previous lecture.

We now provide a slightly different description of this process of patching using the notion of kernel, image and exactness for sheaf functors.

Kernel equivalence relation is a sheaf

Given a morphism $\eta : F \to G$ of sheaf functors **CRing** to **Set**, we have defined

$$E(R) = \{(f, f') \in F(R) : \eta_R(f) = \eta_R(f') \text{ in } G(R)\}$$

This gives a functor E from **CRing** to **Set** with *two* natural transformations $\pi_i : E \to F$ for $i = 1, 2$ corresponding to the two projections.

We can think of $E(R)$ as the "set-theoretic" kernel (or kernel pair) of η_R. We also denote the functor E as $\ker \eta$.

Let us check that E is a sheaf. Note that $E \subset F \times F$ and the latter *is* a sheaf.

Given a ring R and elements $u_1, \ldots, u_k$ generating the unit ideal in R.

Suppose (f_i, f_i') are elements of $E(R_{u_i})$.

The patching condition says that (f_i, f_i') gives the same element as (f_j, f_j') in $E(R_{u_i u_j}) \subset F(R_{u_i u_j})^2$.

There are unique elements f and f' in $F(R)$ which map to f_i and f_i' (respectively) in $F(R_{u_i})$.

Let g (respectively g') be the image of f (respectively f') in $G(R)$.

In order to show that (f, f') lies in $E(R)$, we wish to show that $g = g'$.

By assumption the image g_i of g in $G(R_{u_i})$ is also the image of f_i. Similarly, the image g_i' of g' is also the image of f_i'.

Since (f_i, f_i') lies in $E(R_{u_i})$ we see that $g_i = g_i'$ in $G(R_{u_i})$.

By the sheaf property of G, we see that $g = g'$ is the *unique* element patching the tuple $(g_i) = (g_i')$.

This shows that (f, f') is in $E(R)$ as required. Hence $\ker \eta$ is a sheaf.

Image of a morphism of sheaves

Given a natural transformation (morphism) $\eta : F \to G$ of sheaf functors from **CRing** to **Set** what is the *image* sheaf $\operatorname{im} \eta$?

Naively, one might take the image of $\eta_R : F(R) \to G(R)$ for every ring R.

However, our examples from patching show that more refined approach is required!

Recall how $\mathbb{A}^2 \setminus \{(0,0)\}$ is written as the sheaf-theoretic union of $U_1 = A(x, y, u; ux - 1)$ and $U_2 = A(x, y, v; vy - 1)$ in $\mathbb{A}^2$.

In this case, an R-point of $\mathbb{A}^2 \setminus \{(0,0)\}$ is a pair (a, b) in R^2 such that $\langle a, b \rangle = R$ is the unit ideal in R.

It *need not* be the case that a or b is a *unit* in R.

What we *do* have is that (a, b) gives an R_a-point of U_1 and an R_b-point of U_2.

This is how we express $\mathbb{A}^2 \setminus \{(0,0)\}$ as the *image* of $U_1 \sqcup U_2 \to \mathbb{A}^2$.

This suggests that we declare the *sheaf-theoretic image* $\operatorname{im} \eta$ of a morphism $\eta : F \to G$ of sheaves as follows.

> $\operatorname{im} \eta(R)$ *consists of* R*-points* $g \in G(R)$ *such that there are elements* $u_1, \ldots, u_k$ *of* R *generating the unit ideal in it and points* $f_i \in F(R_{u_i})$ *such that the image of* f_i *in* $G(R_{u_i})$ *is the same as the image of* g*, for* $i = 1, \ldots, k$*.*

Conversely, given $f_i \in F(R_{u_i})$, suppose that $g_{i,j}$ is the image in $G(R_{u_i u_j})$ under the composite $F(R_{u_i}) \to G(R_{u_i}) \to G(R_{u_i u_j})$.

If $g_{i,j} = g_{j,i}$ for all i and j, then the images $g_i \in G(R_{u_i})$ satisfy the patching condition for an R-point of the sheaf G.

Hence, there is a *unique* element $g \in G(R)$ which gives g_i in $G(R_{u_i})$.

We say that the morphism η is *onto* or *surjective* if $\operatorname{im}\eta = G$.

In other words, for *every* $g \in G(R)$:

> *There are elements* $u_1, \ldots, u_k$ *of* R *generating the unit ideal in it and points* $f_i \in F(R_{u_i})$ *such that the image of* f_i *in* $G(R_{u_i})$ *is the same as the image of* g*, for* $i = 1, \ldots, k$*.*

This is the condition for η to be *onto*.

Sheaf quotient by an equivalence relation

Given a sheaf F and a sheaf eqivalence relation $E \subset F \times F$ as above, one can construct the sheaf quotient F/E in a manner similar to the previous lecture.

Given a ring R patching data $(\mathbf{u}, \mathbf{f})$ for an element of $(F/E)(R)$ are given as follows:

- We have $u_1, \ldots, u_k$ elements of R that generate the unit ideal in R.
- We have elements f_i in $F(R_{u_i})$ for each i.
- The pair (f_i, f_j) lies in $E(R_{u_i u_j})$ for each i and j.

We wish to define $(F/E)(R)$ as the quotient of patching data under an equivalence as defined below.

Given *another* set of elements $v_1, \ldots, v_m$ in R such that $\langle v_1, \ldots, v_m \rangle = R$, we note that if we define $w_{i,j} = u_i v_j$, then the collection of $w_{i,j}$ *also* generate the unit ideal in R.

Let $f'_{i,j}$ be the image of f_i via the set map $F(R_{u_i}) \to F(R_{w_{i,j}})$.

We say that $(\mathbf{w}, \mathbf{f}')$ is a refinement of the patching data $(\mathbf{u}, \mathbf{f})$ using the tuple $\mathbf{v}$.

Given two patching data $(\mathbf{u}, \mathbf{f})$ and $(\mathbf{v}, \mathbf{g})$ we can form:

- the refinement $(\mathbf{w}, \mathbf{f}')$ of $(\mathbf{u}, \mathbf{f})$ using the tuple $\mathbf{v}$.
- the refinement $(\mathbf{w}, \mathbf{g}')$ of $(\mathbf{v}, \mathbf{g})$ using the tuple $\mathbf{u}$.

Note that $w_{i,j} = u_i v_j$ are the same in both refinements.

We declare $(\mathbf{u}, \mathbf{f}) \sim (\mathbf{v}, \mathbf{g})$ if $(f'_{i,j}, g'_{i,j})$ lie in $E(R_{w_{i,j}})$ for all i and j.

Exactness

- Given a natural transformation $\eta : F \to G$
- There is an *image* sheaf $\operatorname{im} \eta$ with a factoring of η as $F \to \operatorname{im} \eta \to G$.
- There is a *kernel pair* sheaf $\ker \eta$ with a pair of morphisms $\ker \eta \rightrightarrows F$ which gives an equivalence relation on $F(R)$ for each ring R.

The *quotient* by the equivalence relation can be constructed as was done above.

One checks that the quotient of F by $\ker \eta$ is *precisely* $\operatorname{im} \eta$.

Affine open cover

Given a $\mathbb{Z}$-affine scheme $X = A(x_1, \ldots, x_p; f_1, \ldots, f_q)$, we have defined an affine open subscheme to be a subscheme given by $X_g = A(x_1, \ldots, x_p, v; f_1, \ldots, f_q, vg - 1)$ for some g.

If g and g' are polynomials in $x_1, \ldots, x_p$, that are equal in $\mathcal{O}(X)$, then it is clear that $X_g = X_{g'}$ in a natural way.

Hence, by abuse of notation, we consider g as an element of $\mathcal{O}(X)$.

Given a collection (g_i) of elements of $\mathcal{O}(X)$, the disjoint union $\sqcup_i X_{g_i}$ is called an *affine open cover* of X.

When there are finitely many i (as is usually the case), we have seen that $\sqcup_i X_{g_i}$ is also a $\mathbb{Z}$-affine scheme.

Scheme as a Quotient

As seen in the lecture on patching, a scheme F is a quotient of a $\mathbb{Z}$-affine scheme.

In other words, we are given an *onto* morphism $\eta : X \to F$ of sheaf functors where X is a $\mathbb{Z}$-affine scheme.

In that case, F is the *quotient* of the $\mathbb{Z}$-affine scheme X by the equivalence relation $E = \ker \eta$.

For F to be a scheme, we require some *additional* conditions on E.

1. E should itself be a $\mathbb{Z}$-affine scheme.
2. $E \rightrightarrows X$ is an affine open cover (under both morphisms).

Warning: The above two *conditions* are necessary *but not sufficient* for this to be the description of a scheme.

This was an **error** in the previous version of the slides and in the lecture!

The *additional* condition required is that $E = \sqcup_{i,j=1}^{n} V_{i,j}$ and $X = \sqcup_{i=1}^{n} U_i$ so that the morphism $E \to X \times X$ is made up of $V_{i,j} \to U_i \times U_j$.