Idempotents

Match each ring below with the number of idempotents in it. (Hint: You can use algebra or geometry!)

1. $\mathcal{O}(X)$ where $X = A(x, y;\ xy)$
2. $\{0\}$, the zero ring
3. $\mathcal{O}(X)$ where $X = A(x;\ x(x-1))$
4. $\mathbb{Z}[x_1, \ldots, x_p]$, the polynomial ring in p variables for some non-negative integer p
5. $\mathcal{O}(X)$ where $X = A(w, x, y;\ xy, w(x+y) - 1)$

In general, if we can write the ring R as $R = R_1 \oplus \cdots \oplus R_k$ *as a ring* then there are 2^k idempotents corresponding to taking the element 0 or 1 in each factor; here we are assuming that R_i is *not* the zero ring for any i.

1. The ring $\mathbb{Z}[x, y]/\langle xy\rangle$ has no idempotents other than 0 and 1. Thus there are 2 idempotents.
2. The zero $\{0\}$ has only one element and it is an idempotent. (Strangely, this also corresponds to $k = 0$ in the above formula!)
3. The ring $\mathbb{Z}[x]/\langle x(x-1)\rangle$ is also the sum $[Z][x]/\langle x\rangle \oplus \mathbb{Z}[x]/\langle x-1\rangle$ so it is $2^2 = 4$ idempotents. These are 0, 1, x and $1 - x$.
4. The polynomial ring $\mathbb{Z}[x_1, \ldots, x_p]$ has no idempotents other than 0 and 1. Thus there are 2 idempotents.
5. In the ring $\mathbb{Z}[w, x, y]/\langle xy, w(x+y) - 1\rangle$ the elements xw and yw are non-zero idempotents that add up to 1. In fact, one checks that this is the same ring as $\mathbb{Z}[x, u]/\langle ux - 1\rangle \oplus \mathbb{Z}[y, v]/\langle vy - 1\rangle$. Thus, there are 4 idempotents.

Geometrically, we can argue follows:

1. This case is a pair of *intersecting* lines so it cannot be written as a *disjoint* union of two schemes.
2. This is the empty scheme. So it is special!
3. This is a pair of *distinct* points so it can be written as a disjoint union of the points.
4. The affine space $\mathbb{A}^p$ cannot be written as a disjoint union of two schemes.
5. This is a pair of intersecting lines with the point of intersection *removed*. So it *can* be written as a disjoint union of two schemes.