Sheaves as local homemorphisms

Given a topological space X, let $\mathcal{X}$ be defined as:

- Objects are open sets U in X.
- Given open sets U and V, the set $\mathrm{Mor}(U, V)$ is a singleton $\{i_V^U\}$ if V is a subset of U and empty otherwise.

Define composition in the only possible way!

Q1. Show that with the above definitions, $\mathcal{X}$ is a category.

Solution 1. Let us note that

1. The identity morphism is i_U^U.
2. Given $U \subset V$ and $V \subset W$ are open sets we have $i_V^W \circ i_U^V = i_U^W$.
3. The remaining properties of identity and associative law follow from the fact that if there is a morphism between two objects, it is unique.

Given a continuous map $f : Y \to X$ we say it is a *local homeomorphism* if for every $y \in Y$ there is an open set V in Y such that $f_{|V} : V \to f(V)$ is a homeomorphism for the induced topology on V as a subset of Y and $f(V)$ as a subset of X.

Q2. Show that the map $\exp : \mathbb{C} \to \mathbb{C}$ is a local homeomorphism.

(Hint: Inverse function theorem.)

Solution 2. Note that the morphism at the level of real and imaginary part is given by $(x, y) \mapsto (u, v) = (e^x \cos y, e^x \sin y)$. It follows that the Jacobian matrix of partial derivaties is

$$J(u, v; x, y) = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}$$

The determinant of this matrix is e^{2x} which is never 0.

By the inverse function theorem, this means that for each (x_0, y_0), there is a neighbourhood U of $(u_0, v_0) = (e^{x_0} \cos y_0, e^{x_0} \sin y_0)$ and a pair of differentiable functions (f, g) of (u, v) in the open set U such that

$$(f(e^x \cos y, e^x \sin y), g(e^x \cos y, e^x \sin y)) = (x, y)$$

and $(f(u_0, v_0), g(u_0, v_0)) = (x_0, y_0)$. It follows that the given map exp has a continuous (even differentiable!) local inverse and is thus a local homeomorphism.

Given a continuous map $f : Y \to X$ and an open set U in X, a *section* of f over U is a continuous map $s : U \to Y$ such that $f \circ s : U \to X$ is the inclusion of U in X.

Q3. Given a pair $V \subset U$ of open subsets of X and a section s of f over U, show that the restriction $s_{|V}$ of s to V is a section of f over V.

Solution 3. The restriction $s_{|V}$ of a continuous map is also continuous. Since $f(s(x)) = x$ for all s in U, it also follows for all x in V.

For each U define $F(U)$ to be the set of sections of f over U. For a section s of f over U and for a open subset V of U define $F(i_V^U) : F(U) \to F(V)$ by the equation $F(i_V^U)(s) = s_{|V}$.

Q4. Show that F defines a *contravariant* functor from $\mathcal{X}$ to **Set**. (**Note**: The word "contravariant" was missed in the original question statements for this question and later questions!)

Solution 4. Suppose $U \subset V \subset W$ is a sequence of open subsets. Given $s \in F(W)$, we have

$$F(i_U^W)(s) = s_{|U} = (s_{|V})_{|U} = F(i_U^V)(F(i_V^W))$$

Hence, F is a contravariant functor from $\mathcal{X}$ to $\mathbf{S}$.

Given a *contravariant* functor G from $\mathcal{X}$ to **Set**.

Given open sets U_i for each i in some indexing set I, a collection $s_i \in G(U_i)$ is said to satisfy *patching* if, for each i and j in I we have

$$G(i_{U_i \cap U_j}^{U_i})(s_i) = G(i_{U_i \cap U_j}^{U_j})(s_j)$$

We say that G is a *sheaf* if for every collection satisfying patching, there is a unique s in $G(U)$ such that $s_i = G(i_{U_i}^U)(s)$, where $U = \cup_i U_i$.

Q5. Check that F is a sheaf.

Solution 5. A function $s : U \to Y$ is *uniquely* determined by the collection $s_{|U_i}$ since $U = \cup_i U_i$.

Conversely, given functions $s_i : U_i \to Y$, such that $(s_i)_{|U_i \cap U_j} = (s_j)_{|U_i \cap U_j}$, there is a *unique* function $s : U \to Y$ such that $s_{|U_i} = s_i$.

If s_i are continuous, then s is *also* continuous since continuity is a *local* property; it is enough to check that *small* enough open sets have open inverse images and we know this properties for open sets *contained* in at least one U_i.

Given a sheaf functor G from $\mathcal{X}$ to **Set** and a point x in X, we define G_x to be the equivalence classes of pairs (s, U) as follows:

- U is an open set containing x and $s \in G(U)$, and
- $(s, U) \sim (t, V)$ is there is an open set W satisfying $x \in W \subset U \cap V$ such that $G(i_W^U)(s) = G(s_W^V)(t)$.

Let us define Y to be the disjoint union of G_x as x varies over X and $g : Y \to X$ be the map which sends all of G_x to x.

For an open set U in X and an element s in $G(U)$ and $x \in U$, define $\tilde{s}(x)$ to the equivalence class in G_x of (s, U). This defines a map $\tilde{s} : U \to Y$.

Q6. Show that $g \circ \tilde{s}$ is the natural inclusion of U in X.

Solution 6. The equivalence class of an element (s, U) of G_x is sent to x under g. If x is an element of U, then the $\tilde{s}(x)$ is the equivalence class of (s, U) in G_x. Hence, its image is x.

Given open subsets U and V in X such that $V \subset U$ and an element $s \in G(U)$, let $r = G(i_V^U)(s)$.

Q7. With notation as in Q6, Show that $\tilde{r}$ is the same as $\tilde{s}_{|V}$.

Solution 7. Given x in V, we note that $\tilde{r}(x)$ is the equivalence class of (r, V) in G_x. Since $r = s_{|V}$, we see that this is the same as the equivalence class of (s, U) in G_x. Hence, we see that $\tilde{r}(x) = \tilde{s}(x)$ for x in V.

Given an open subset U of X and an element s in $G(U)$, let $U_s = \tilde{s}(U)$ considered as a subset of Y, with notation as in Q6.

Q8. Show that, as U and s vary, the sets U_s give a basis for a topology on Y.

Solution 8. Suppose that the equivalence class of (t, W') in G_z is in the intersection $U_s \cap V_r$ for s in $G(U)$ and r in $G(V)$. This means that z lies in $W = U \cap V \cap W'$.

The following equivalence classes in G_z are all the *same* under the above hypothesis.

$$(s_{|W}, W) \sim (r_{|W}, W) \sim (t, W') \sim (t_{|W}, W)$$

It follows that $W_{t_{|W}}$ is a subset of $U_s \cap W_r$ and contains this point of G_z.

This shows that the given collection satisfies the property of a basis for a topology.

Q9. With topology on Y as defined above, show that $g : Y \to X$ is a local homemorphism.

Solution 9. Given a point of G_z, it is the equivalence class of some element of the form (s, U). One checks that the set U_s maps to U under g.

Since g is a bijection on basic open sets, it is a local homeomorphism.

Thus, the notion of sheaf functor from $\mathcal{X}$ to **Set** and the notion of local homemorphism $f : Y \to X$ coincide.

Given topological spaces X and Y, for every open set U in X, let $Y_\cdot(U)$ be the set of continuous maps $s : U \to Y$ (in the topology on U induced from X).

For i_V^U a morphism in $\mathcal{X}$ as in Q1 and $s : U \to Y$, let $Y_\cdot(i_V^U)(s) = s_{|V}$ be the restrction of s to V.

Q10. Show that $Y_\cdot$ is a sheaf functor $\mathcal{X}$ to **Set**.

Solution 10. If we examine the solution to Q5 we see that this was already proved there!

If $U = \cup_i U_i$ is a union of open sets and $s_i : U_i \to Y$ are given such that $(s_i)_{|U_i \cap U_j} = (s_j)_{|U_i \cap U_j}$, then we can define $s : U \to Y$ by defining it as $s(x) = s_i(x)$ for $x \in U_i$.

Secondly, since s_i is continuous for each i and continuity is a local property, we see that s is continuous.

In particular, note that the sheaf $\mathbb{R}_\cdot$ represents continuous real-valued functions on (open sets of) X.